Zariski Topology

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1 Introduction

Let k be a field. In algebraic geometry, there is an obvious emphasis on algebraic subsets of affine space $\mathbb{A}^n(k)$ (which we will abbreviate as $\mathbb{A}^n$ when the underlying field is unimportant). Arguably, these are among the central objects of study. One of the first results in classical algebraic geometry is that the property of a subset that is to be algebraic is closed under certain set-theoretic operations. In particular,

- If $I, J \subseteq k[X_1, \ldots, X_n]$, then it is clear that $V(I) \cup V(J) = V(\{FG \colon F \in I, G \in J\})$. This implies that the collection of algebraic sets is closed under finite unions.

- If A is any nonempty set such that for every $\alpha \in A$ we have $I_\alpha \subseteq k[X_1, \ldots, X_n]$, then $\bigcap_{\alpha \in A} V(I_\alpha) = V\left(\bigcup_{\alpha \in A} I_\alpha\right)$. Hence, the collection of algebraic sets is closed under arbitrary intersections.

This observation shows us that the algebraic subsets of $\mathbb{A}^n$ can be said to induce a *topology* on $\mathbb{A}^n$.

Definition: Let X be a nonempty set. A **topology on** X is a collection of subsets $\tau \subseteq \mathscr{P}(X)$ such that $\varnothing, X \in \tau$ and τ is closed under arbitrary unions and finite intersections. A subset $U \subseteq X$ is said to be **open** if and only if $U \in \tau$ and **closed** if and only if $X \setminus U \in \tau$. The pair (X, τ) is a **topological space**.

Since the algebraic subsets of $\mathbb{A}^n$ are closed under finite unions and arbitrary intersections, it is clear that they form the closed sets with respect to a topology on $\mathbb{A}^n$. This topology is what we will study.

Definition: The **Zariski topology on** $\mathbb{A}^n$ is the topology on $\mathbb{A}^n$ that takes the closed sets to be the algebraic subsets of $\mathbb{A}^n$. That is, it is the topology on $\mathbb{A}^n$ consisting of the complements of algebraic subsets of $\mathbb{A}^n$.

This topology encodes the information contained in the collection of algebraic sets of an affine space. Throughout this discussion, we will need various notions from topology.

Definition: Let (X, τ) be a topological space and $x \in X$. A **neighborhood of** x is an open set U containing x.

Definition: Let (X, τ) be a topological space, and let $U \subseteq X$. The intersection of all closed sets containing U is the **closure** of U, denoted $\overline{U}$. It is the smallest closed set containing U.

Definition: Let (X, τ) be a topological space. A set $U \subseteq X$ is said to be **dense in** X if $\overline{U} = X$.

Definition: Let (X, τ) and (Y, σ) be topological spaces. A map $f \colon X \to Y$ is **continuous** if the preimage of every open set in Y is an open set in X. A continuous bijection with a continuous inverse is a **homeomorphism**.

Definition: A topological space (X, τ) is said to be **Hausdorff** if for every $x, y \in X$ with $x \neq y$, there exists a neighborhoods U of x and V of y such that U and Y are disjoint.

Definition: Let (X, τ) be a topological space and $V \subseteq X$. V is said to be **compact** if every open cover of V admits a finite subcover, that is, for every nonempty set A with $U_\alpha \in \tau$ for every $\alpha \in A$ and $V \subseteq \bigcup_{\alpha \in A} U_\alpha$, there exists $\alpha_1, \ldots, \alpha_n \in A$ such that $V \subseteq \bigcup_{i=1}^n U_{\alpha_i}$. V is said to be **precompact** if $\overline{V}$ is compact.

Definition: Let X be a nonempty set. The collection of subsets of X whose complements are finite forms a topology on X, called the **cofinite topology on** X.

Definition: Let X be a nonempty set. The topology given by $\mathscr{P}(X)$ is the **discrete topology on** X.

Definition: Let X be a nonempty set and let τ and σ both be topologies on X such that $\tau \subseteq \sigma$. We say that τ is **coarser** than σ and σ is **finer** than τ.

Definition: Let (X, τ) be a topological space. If there exists a metric d on X such that d induces the topology τ, we say that τ (or X when the topology is understood) is **metrizable**.

2 Basic Properties

We will first study the Zariski topology in the simplest cases. It turns out that the Zariski topology has especially simple characterizations in the case that k is a finite field and in the case of affine 1-space, $\mathbb{A}^1$.

Proposition 1: *If k is a finite field, the Zariski topology on $\mathbb{A}^n(k)$ coincides with the discrete topology.*

Proof. This is equivalent to the statement that every subset of $\mathbb{A}^n(k)$ is algebraic. Indeed, every singleton is trivially algebraic and every subset of $\mathbb{A}^n(k)$ can be written as a finite union of singletons. $\square$

Proposition 2: *The Zariski topology on $\mathbb{A}^1$ coincides with the cofinite topology.*

Proof. This is equivalent to the statement that a subset of $\mathbb{A}^1$ is algebraic if and only if it is finite or otherwise all of $\mathbb{A}^1$. Suppose $S \subseteq \mathbb{A}^1$ is algebraic. Then $S \subseteq V(f)$ for some $f \in k[X]$. Suppose further that $S \neq \mathbb{A}^1$ so that f can be chosen to be a nonzero polynomial. Now by the factor theorem, $|S| \leq V(F) \leq \deg f < \infty$. In the other direction, suppose that S is either finite or all of $\mathbb{A}^1$. If S is all of $\mathbb{A}^1$, then S is trivially algebraic. Otherwise, if S is finite, then we may write $S = V(f)$ where $f = \prod_{s \in S} (X - s) \in k[X]$. $\square$

Proposition 1 says that the algebraic sets over a finite field are uninteresting in the sense that *every* subset is such a set. In this extreme case, the topology is so fine that it fails to significantly diverge from our intuition. We will see in Proposition 5 and Proposition 6 that the Zariski topology behaves much more strangely over infinite fields.

The Zariski topology gives us a way to topologically interpret the operations V and I from classical algebraic geometry.

Proposition 3: *Let $S \subseteq \mathbb{A}^n$. $V(I(S)) = \overline{S}$.*

Proof. Of course, $I(S)$ consists of all polynomials vanishing on S so $V(I(S))$ is a closed set containing S. Hence, $\overline{S} \subseteq V(I(S))$. On the other hand, if $x \in V(I(S))$, then every polynomial vanishing on S will also vanish on x. Hence, if $S \subseteq V(J)$, then $x \in V(J)$ because every polynomial in J must vanish on x. That is, x is in every closed set containing S, so $x \in \overline{S}$. This completes the last inclusion. $\square$

An interesting question is that of the relationship between the standard Euclidean topology on $\mathbb{R}^n$ or $\mathbb{C}^n$ (where open sets are defined in the obvious way with the Euclidean metric) and the Zariski topology on $\mathbb{A}^n(\mathbb{R})$ or $\mathbb{A}^n(\mathbb{C})$. The first direct relationship is that the Zariski topology is strictly coarser than the Euclidean topology.

Proposition 4: *Let $k \in \{\mathbb{R}, \mathbb{C}\}$. The Zariski topology on $\mathbb{A}^n(k)$ is strictly coarser than the Euclidean topology on k^n.*

Proof. First we show that the Zariski topology is coarser than the Euclidean topology by showing that every closed set in the Zariski topology is also closed in the Euclidean topology. Let S be a closed set in the Zariski topology. Then we can write $S = V(I)$ for some $I \subseteq k[X_1, \ldots, X_n]$. Pick $f \in I$. We can interpret f as a polynomial map $f \colon k^n \to k$. Since f is polynomial, this map is continuous with respect to the Euclidean topologies on k^n and k. In particular, $\{0\} \subseteq k$ is closed in the Euclidean topology, so $f^{-1}(\{0\}) \subseteq k^n$ is closed in the Euclidean topology. Now, $S = V(I) = \bigcap_{f \in I} f^{-1}(\{0\})$ is closed in the Euclidean topology as an intersection of closed sets.

Now we will check that Zariski topology is strictly coarser by finding a set that is closed in the Euclidean topology but not the Zariski topology. Let U be a nonempty subset of k^n that is open and not dense with respect to the Euclidean topology (one can choose, for instance, an open ball). Let $V = \overline{U}$, where the closure is taken with respect to the Euclidean topology. V is a closed set in the Euclidean topology. Notice further that since U is not dense, $V \neq \mathbb{A}^n(k) = k^n$. We claim that V is not an algebraic set.

Suppose to the contrary that V is algebraic. Then $V = V(I)$ for some $I \subseteq k[X_1, \ldots, X_n]$. Pick $f \in I$. Since $U \subseteq V$, f must vanish on U. If $n = 1$, then f is polynomial and thus an analytic function of one variable that is zero on an open set (in the Euclidean topology). By the identity theorem from analysis, f is identically zero. If $n > 1$, for every $y = (y_1, \ldots, y_{n-1}) \in \mathbb{A}^{n-1}(k)$, define $g_y \in k[X]$ by $g_y(X) = f(y_1, \ldots, y_{n-1}, X)$. Now we invoke the following topological fact: the projection map $\pi_n \colon k^n \to k$ given by $\pi(x_1, \ldots, x_n) = x_n$ is an open map.[1] Hence, $\pi_n(U)$ is open with respect to the Euclidean topology. So for every $y \in \mathbb{A}^{n-1}(k)$, g_y vanishes on the open set $\pi_n(U)$ of k, so g_y is identically zero for every y by the identity theorem. Since y is arbitrary, we conclude in this case as well that f is identically zero.

Hence, the only function in I must be the constant zero function, and so $V = V(I) = \mathbb{A}^n(k)$, contradiction. It follows that V is not algebraic. □

Proposition 4 manifests itself in a concrete way. The Euclidean topology is Hausdorff (in fact, the Euclidean topology obeys much stronger separation axioms). However, the Zariski topology is so much coarser than the Euclidean topology that it fails to be Hausdorff (on $\mathbb{R}^n$ or $\mathbb{C}^n$). We can state this more generally.

Proposition 5: *If k is an infinite field, then $\mathbb{A}^n(k)$ with the Zariski topology is not Hausdorff.*

Proof. Let k be an infinite field and let $x, y \in \mathbb{A}^n(k)$ be distinct points. Suppose there exist disjoint neighborhoods U and V of x and y, respectively. Since U^c and V^c are both closed, there exists sets $S, T \subseteq k[X_1, \ldots, X_n]$ such that $U^c = V(S)$ and $V^c = V(T)$. The condition that U and V are disjoint is then equivalent to $V(S)^c \cap V(T)^c = \varnothing$. Therefore, there exists no point $P \in \mathbb{A}^n(k)$ such that $P \notin V(S)$ and $P \notin V(T)$. That is, every point P is either in $V(S)$ or $V(T)$. That is, at any $P \in \mathbb{A}^n(k)$, either all $f \in S$ vanish or all $g \in T$ vanish.

Suppose that the only polynomial in S is the zero polynomial. Then, $U^c = V(S) = \mathbb{A}^n(k)$ so that $U = \varnothing$, contradicting our assumption that $x \in U$. Hence, S contains a nonzero polynomial F. Let $g \in T$ be arbitrary. gF is a polynomial. Since at any point, either all polynomials in S or all polynomials in T vanish, gF vanishes on all of k. Because k is infinite, gF must be the constant zero polynomial.[2] Since the polynomial ring $k[X_1, \ldots, X_n]$ is an integral domain and F is not the zero polynomial by construction, g must be the zero polynomial. That is, the only polynomial in T is the zero polynomial, and we arrive at a contradiction as before. □

Note that the hypothesis in Proposition 5 that k is infinite is crucial, though it is used in a rather subtle way. If k is finite, Proposition 1 says that the Zariski topology is the discrete topology, which is clearly Hausdorff.

One interesting consequence of Proposition 5 is that the Zariski topology is not metrizable, because every metric space has a Hausdorff topology.

While the Zariski topology is not Hausdorff in general, it does always satisfy the weaker separation axiom of T_1. That is, for any pair of distinct points $x, y \in \mathbb{A}^n$, there exists a neighborhood U of x but not of y. This neighborhood is easy to construct: if $y = (y_1, \ldots, y_n)$, then we may select $U = V(X_1 - y_1, \ldots, X_n - y_n)^c$.

The failure of the Zariski topology to be Hausdorff in general is one of the significant departures from the more intuitive properties of the Euclidean topology. It turns out that it gets worse.

Proposition 6: *If k is an infinite field, every nonempty open subset of $\mathbb{A}^n(k)$ with the Zariski topology is dense.*

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[1] This is not a hard thing to see; it follows directly from the definition of product topology, which we will not define here.

[2] This is by induction on n and the fact that in one variable, the number of roots of a polynomial is bounded by the degree of the polynomial, which is a finite number. This is problem 1.4 from Fulton, assigned in Homework 1.

Proof. Let $U \subseteq \mathbb{A}^n$ be open and nonempty. Note that $\mathbb{A}^n(k) = \overline{U} \cup U^c$ is a union of closed sets. Because k is an infinite field, $\mathbb{A}^n(k)$ is irreducible,[3] so either $\overline{U} = \mathbb{A}^n(k)$ or $U^c = \mathbb{A}^n(k)$. But the latter is ruled out since U is nonempty. $\square$

Of course this result patently false in the Euclidean topology. One may take, for instance, any bounded set that is open with respect to Euclidean topology and see that such a set will not be dense.

We demonstrate one last way in which the Zariski topology deviates spectacularly from the Euclidean topology.

Proposition 7: *$\mathbb{A}^n$ with the Zariski topology is compact.*

Proof. Let $\bigcup_{\alpha \in A} U_\alpha = \mathbb{A}^n$ be an open cover. We wish to extract a finite subcover, so it suffices to assume without loss of generality that A is infinite. Consider an arbitrary countable subset $\{\alpha_j\}_{j \in \mathbb{N}} \subseteq A$ and define $V_j = \bigcup_{i=1}^{j} U_{\alpha_i}$ so that the open sets V_j form an ascending chain

$$V_1 \subseteq V_2 \subseteq \dots$$

so that the complements form the descending chain

$$V_1^c \supseteq V_2^c \supseteq \dots .$$

The sets V_j^c are closed so we can pick ideals $I_1, I_2, \dots \subseteq k[X_1, \dots, X_n]$ where k is underlying field, such that $V_j^c = V(I_j)$ for all $j \in \mathbb{N}$. Since the V operation is order-reversing, the ideals I_j form the ascending chain

$$I_1 \subseteq I_2 \subseteq \dots$$

By the Hilbert basis theorem, $k[X_1, \dots, X_n]$ is a Noetherian ring, so there exists an integer N such that $I_j = I_{j+1}$ for all $j \geq N$.[4] It follows that $V_j = V_{j+1}$ for all $j \geq N$. In particular, V_N is an upper bound to our chain of V_j ordered by inclusion. Hence, when we partially order the collection

$$\mathscr{C} = \left\{ \bigcup_{i=1}^{j} U_{\beta_i} : j \in \mathbb{N}, \{\alpha_i\}_{i \in \mathbb{N}} \subseteq A \right\}$$

by inclusion, every increasing chain has an upper bound, so by Zorn's lemma, there exists a maximal element U of $\mathscr{C}$. Suppose $U \neq \mathbb{A}^n$. Then, there exists $x \in \mathbb{A}^n \setminus U$ and $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. But by definition, $U = \bigcup_{i=1}^{m} U_{\beta_i}$ for some sequence $\{\beta_i\}_{i \in \mathbb{N}} \subseteq A$. We may modify this sequence so that $\gamma_i = \beta_i$ for all $i \neq N+1$ and $\gamma_{N+1} = \alpha_0$. Then, $\bigcup_{i=1}^{N+1} U_{\gamma_i} \in \mathscr{C}$ and properly contains U, contradicting its maximality. Hence, we have found the finite subcover

$$\bigcup_{i=1}^{m} U_{\beta_i} = U = \mathbb{A}^n.$$

$\square$

If one is uncomfortable with the use of Zorn's lemma above, one can prove that $\mathbb{A}^n$ is countably compact (that is, every countable open cover has a finite subcover) by adapting the main portion of the proof above without needing to invoke Zorn's lemma.[5] Indeed, the Euclidean topology is not compact or even countably compact.

[3] This is Problem 1.29 from Fulton, assigned in Homework 2. The solution depends on the fact that over an infinite field, a polynomial that vanishes everywhere is the zero polynomial (which is the content of the previous footnote).

[4] In MATH 106, we define a Noetherian ring to be a ring whose ideals are finitely generated. More generally, a Noetherian module is defined to be module whose submodules satisfy the so-called ascending chain condition: every ascending chain of submodules becomes constant. This is equivalent to the condition that every submodule is finitely generated.

[5] Incidentally, one can also prove compactness using the weaker axiom of countable choice.

3 The Cayley-Hamilton Theorem

While the properties of the Zariski topology are interesting, it is somewhat obscure what can be done with it. By using the collection of algebraic sets to form the Zariski topology, we passed from algebra to topology. This can be reversed: we may use the topological properties we developed in the previous section to say something about algebra. The Cayley-Hamilton theorem is a well-known theorem from linear algebra. As it turns out, it may be proven by considering the Zariski topology on affine space.

Theorem (Cayley-Hamilton Theorem): *Let k be an infinite field and $A \in M_n(k)$ an $n \times n$ matrix. A satisfies its own characteristic equation.*

Proof. Let $\overline{k}$ be the algebraic closure of k. Observe that the coefficients of χ_A are in $k \subseteq \overline{k}$ for any $A \in M_n(k)$. So it suffices to establish the theorem for $\overline{k}$.

We identify $M_n(\overline{k})$ with $\mathbb{A}^{n^2}(\overline{k})$. Let

$$X = \left\{A \in M_n(\overline{k}) \colon \chi_A(A) = [0]\right\},$$

$$U = \left\{A \in M_n(\overline{k}) \colon A \text{ has } n \text{ distinct eigenvalues}\right\}.$$

U is clearly nonempty, since any diagonal matrix with distinct elements on the diagonal is in U. Pick $A \in U$. Since A has n distinct eigenvalues, there exists an eigenbasis $v_1, \dots, v_n$ of $\overline{k}^{n^2}$ over which A becomes diagonal. Let $\lambda_1, \dots, \lambda_n$ be the distinct eigenvalues of A. Then, the characteristic polynomial factors as $\chi_A(x) = \prod_{i=1}^n (x - \lambda_i)$ so that $\chi_A(A) = \prod_{i=1}^n (A - \lambda_i I)$.

Let $v \in \overline{k}^{n^2}$ be arbitrary. Pick $c_1, \dots, c_n$ so that $v = \sum_{j=1}^n c_j v_j$. Then

$$\chi_A(A)(v) = \prod_{i=1}^n (A - \lambda_i I) \sum_{j=1}^n c_j v_j = \sum_{j=1}^n \left[\left(c_j \prod_{i=1}^n (\lambda_j - \lambda_i)\right) v_j\right] = \sum_{j=1}^n (0 \cdot v_j) = \vec{0},$$

where the second equality comes from the fact that $A - \lambda_i I$ is a diagonal matrix in the basis we have chosen. Hence $\chi_A(A) = [0]$ and $A \in X$. This shows that $U \subseteq X$.

For any $T \in M_n(\overline{k})$, we can consider the discriminant of the characteristic equation of T which is $\Delta(\chi_T) = \prod_{i \neq j} (\lambda_i - \lambda_j)$, where $\lambda_1, \dots, \lambda_n$ is a list of the eigenvalues of T counted with multiplicity. Since $\overline{k}$ is algebraically closed, there is no matrix in $M_n(\overline{k})$ with less than n eigenvalues when counted with multiplicity. Therefore, $A \in U$ if and only if the eigenvalues of A are distinct. This happens if and only if $\Delta(\chi_A) \neq 0$ by construction. So we may alternatively make the characterization

$$U = \left\{A \in M_n(\overline{k}) \colon \Delta(\chi_A) \neq 0\right\}.$$

It is a fact that $\Delta(\chi_A)$ is a polynomial in the coefficients of χ_A,[6] which is itself a polynomial in the entries of A, so there exists a polynomial $f \in \overline{k}\left[X_1, \dots, X_{k^{n^2}}\right]$ such that $\Delta(\chi_A) = f(A)$. Clearly, $\{0\}$ is algebraic, so $f^{-1}(\{0\})$ is algebraic as the preimage of an algebraic set under a polynomial map,[7] and $U = \mathbb{A}^{n^2}(\overline{k}) \backslash f^{-1}(\{0\})$ is open in the Zariski topology.

For any $T \in M_n(\overline{k})$, each component of the matrix $\chi_T(T)$ is a polynomial in the entries of T, so the condition $\chi_A(A) = [0]$ is satisfied at the intersection of of the vanishing sets of the polynomials corresponding to each component of $\chi_A(A)$. That is, X is an algebraic set and thus closed in the Zariski topology. We have already shown that $X \supseteq U$. Since X is closed, we also have $X \supseteq \overline{U}$. Finally, by Proposition 6, since $\overline{k}$ is infinite and U is nonempty and open, U is dense and we have

$$X \supseteq \overline{U} = \mathbb{A}^{n^2}(\overline{k}).$$

That is, $X = M_n(\overline{k})$.

□

[6] This follows from the fact that $\chi_A(A)$ is a symmetric polynomial in the λ_i by definition, so it is a polynomial in the elementary symmetric sums of the λ_i which are precisely the coefficients of $\chi_A(A)$ (up to a constant) due to Vieta's formulas. Alternatively, this can be shown using Galois theory.

[7] This is the first part of Problem 2.7 in Fulton.

The Cayley-Hamilton theorem in a vacuum is quite mysterious, but the manner of proof above makes it especially interesting. What we have really shown is that if k is an infinite algebraically closed field, when we endow the space of linear operators on k^n with the Zariski topology by identifying it with affine n^2-space, the subset of operators with distinct eigenvalues is dense and thus by Proposition 7, precompact. This is reminiscent of the Arzelà-Ascoli theorem from analysis, which states a certain subset of the space of continuous functions is precompact with respect to a natural topology.

The Zariski topology is perhaps not the natural topology on the space of linear operators on k^n, but this analogy with the Arzelà-Ascoli theorem motivates us to ask even more interesting questions. The power of the Arzelà-Ascoli theorem is that the natural ambient topology in consideration is metrizable, so compactness in the space of continuous functions is equivalent to sequential compactness. In practice, the sequential compactness implied by Arzelà-Ascoli is what is really used. While we already know by Proposition 5 that the Zariski topology is not metrizable, I do not know if affine space with the Zariski topology is sequentially compact.

Question: *Is $\mathbb{A}^n$ with the Zariski topology sequentially compact?*

4 The Spectrum of a Ring

The Zariski topology we have discussed thus far is a topology on affine space over a field. It turns out that this topological space corresponds to another that is quite different.

Definition: Let R be a ring. The collection of prime ideals of R, denoted by $\operatorname{Spec} R$, is the **spectrum of** R.

We can endow the spectrum of a ring with a topology, also called the Zariski topology.

Definition: Let R be a ring. Let $E \subseteq R$. We denote the set of prime ideals of R containing E by $\widetilde{V}(E)$. The subsets of $\operatorname{Spec} R$ of the form $\widetilde{V}(E)$ are the closed sets of the **Zariski topology on** $\operatorname{Spec} R$.

The Zariski topology on the spectrum of a ring is very interesting to study, and many of its properties are shared with the Zariski topology on affine space. There is in fact a correspondence with the Zariski topology on affine space. However, this correspondence is somewhat difficult to state, lengthy to develop, and hard to prove. To conclude this burgeoning expository paper, I will establish a relationship between the operation V that we are familiar with from MATH 106 and $\widetilde{V}$ defined above.

Proposition 8: *Let k be a field and let $R = k[X_1, \ldots, X_n]$. For any ideal $I \subseteq k[X_1, \ldots, X_n]$,*

$$V(I) = V\left(\bigcap_{J \in \widetilde{V}(I)} J\right).$$

Proof. Let $I \subseteq k[X_1, \ldots, X_n]$ be an ideal. It suffices to show that $\bigcap_{J \in \widetilde{V}(I)} J = \sqrt{I}$.[8]

Put $S = \bigcap_{J \in \widetilde{V}(I)} J$. Let $x \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $x^n \in I \subseteq S$. For every $J \in \widetilde{V}(I)$, we have that J is prime and $x^n \in J$, so inductively, $x \in J$. Hence $x \in S$ establishing that $\sqrt{I} \subseteq S$.

In the other direction, suppose now that $x \notin \sqrt{I}$. Let Ω be the collection of all ideals of $k[X_1, \ldots, X_n]$ that do not contain x^n for any $n \in \mathbb{N}$. Ω is nonempty because $\{0\} \in \Omega$. We may partially order Ω by inclusion. Consider a chain of elements of Ω,

$$J_1 \subseteq J_2 \subseteq \ldots.$$

Since this is an ascending sequence of ideals, $\bigcup_{i=1}^{\infty} J_i$ is an ideal. Moreover, no power of x is in the union by construction. Hence $\bigcup_{i=1}^{\infty} J_i$ is an upper bound of the chain and by Zorn's lemma, there exists some maximal element $P \in \Omega$. Since P is not the full ring, there exist $a, b \in k[X_1, \ldots, X_n] \setminus P$. Since P is maximal in Ω, the ideal $P + \langle ab \rangle$ is not in Ω, so in particular $P + \langle ab \rangle \neq P$. Hence, $ab \notin P$. This establishes that P is a prime ideal. Since we have found a prime ideal containing I but not x, we have that $x \notin S$. Hence by contrapositive, $S \subseteq \sqrt{I}$.

∎

[8] This is due to Problem 1.20 in Fulton.

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