Algebraic Topology

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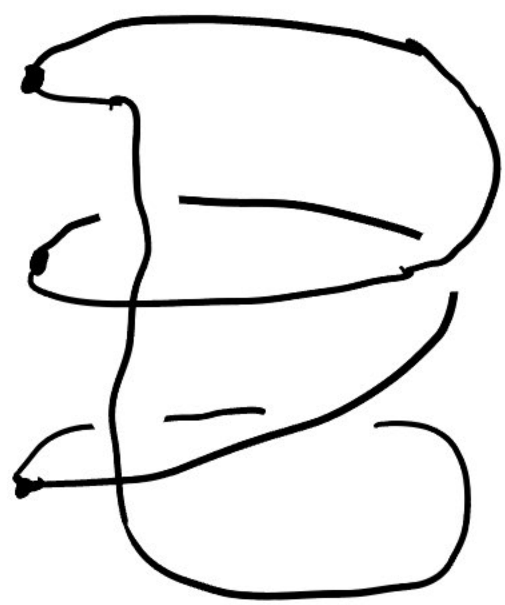

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Part I

Algebraic Preliminaries

Chapter 1

Categories

The mathematics we will do will rest firmly within set theory. More precisely, we will work in a conservative extension of Zermelo-Fraenkel-choice set theory (ZFC) that permits a notion of *classes*, namely von Neumann-Bernays-Gödel set theory (NBG). While we will not be precise about all of the details of our set-theoretic foundations, we will note that classically the set-theoretic point of view tends to emphasize the sets that we are working with. As we will quickly realize, it is worthwhile to care not just about our spaces, but also about the maps between them. *Category theory* offers a systematic way for maps to be put on equal footing with spaces. Therefore, statements that would be cumbersome to express purely set-theoretically often will become much more succinct when expressed categorically.

Algebraic topology is actually the historical impetus for the invention of category theory, so its importance to us cannot be understated.

1.1 Diagrams

We begin with the definition of categories. Note that since our foundations are set-theoretic, we have access to a notion of equality. If our foundations did not allow for a notion of equality, then adding the structure of an equality relation to the category would turn the category into a *strict category*. Since our foundations already include a notion of equality, all of our categories are automatically strict categories, so we have no need for distinguishing between categories and strict categories. We will thereby never mention this again and freely write equations.

Definition 1.1.0.1. A **category** $\mathcal{C}$ consists of the data of two classes, $\operatorname{ob}\mathcal{C}$ (consisting of **objects**) and $\operatorname{mor}\mathcal{C}$ (consisting of **morphisms**), satisfying the following rules.

(1) To each morphism $f \in \operatorname{mor}\mathcal{C}$, we associate an object $\operatorname{dom} f \in \operatorname{ob}\mathcal{C}$ called the **domain** of f and an object $\operatorname{codom} f \in \operatorname{ob}\mathcal{C}$ called the **codomain** of f. For each pair $X, Y \in \operatorname{ob}\mathcal{C}$ we define the **morphism-class** $\operatorname{mor}(X, Y)$ (also denoted $\operatorname{mor}_{\mathcal{C}}(X, Y)$ when we need to emphasize that we are looking at morphisms of the category $\mathcal{C}$) which is the collection of all morphisms $f \in \operatorname{mor}\mathcal{C}$ with $\operatorname{dom} f = X$ and $\operatorname{codom} f = Y$. Notationally, we write $f\colon X \to Y$ to mean $f \in \operatorname{mor}(X, Y)$.

(2) For any pair of morphisms $f, g \in \operatorname{mor}\mathcal{C}$, if $\operatorname{dom} f = \operatorname{codom} g$ then we say that f and g are **composable** and we associate to this composable pair a morphism $f \circ g$ which we call the **composition** of f and g. This composition operation is associative. That is, for any three morphisms $f, g, h \in \operatorname{mor}\mathcal{C}$ such that $\operatorname{dom} f = \operatorname{codom} g$ and $\operatorname{dom} g = \operatorname{codom} h$, we have that $(f \circ g) \circ h = f \circ (g \circ h)$. Hence, it is well-defined to drop parentheses when composing a finite collection of morphisms.

(3) For every object $X \in \operatorname{ob}\mathcal{C}$, there exists an **identity morphism** $\mathbf{1}_X\colon X \to X$ such that for any morphisms $f\colon X \to Y$ and $g\colon Z \to X$ we have $f \circ \mathbf{1}_X = f$ and $\mathbf{1}_X \circ g = g$. $\diamond$

It is clear from Definition 1.1.0.1 that a category encodes the data of a collection of objects and things that behave like functions between them. Indeed, the category Set, which is the category whose class of objects is the class of all sets and whose class of morphisms is the class of all functions between sets, is the prototypical example of a category. However, as we will see, the objects of a category need not be sets at all, and morphisms need not be functions.

Note that we insisted that the objects and the morphisms of a category form classes. Indeed, the objects and morphisms need not form sets (in our foundations they may be classes; in other foundations they could be different things) and thus categories can be "large". In fact, we say that a category $\mathcal{C}$ is *small* if $\operatorname{ob}\mathcal{C}$ and $\operatorname{mor}\mathcal{C}$ are both sets and that $\mathcal{C}$ is *large* otherwise. If every morphism-class of $\mathcal{C}$ is a set, then we say that $\mathcal{C}$ is *locally small*. By the axiom of union in ZFC, a locally small category is small if and only if the objects form a set. In fact, morphism-classes can get quite small. A category in which the morphism-classes contain at most one morphism is called *posetal*. If an object $X \in \operatorname{ob}\mathcal{C}$ is such that for every $Y \in \operatorname{ob}\mathcal{C}$, the morphism-class $\operatorname{mor}(X, Y)$ is a singleton then we say that X is an *initial object*. If for every $Y \in \operatorname{ob}\mathcal{C}$, the morphism-class $\operatorname{mor}(Y, X)$ is a singleton then we say that X is a *final* or *terminal object*. An object that is both final and initial is called a *zero object*.

The definition of categories consists of stringent axioms on the morphisms. This immediately suggests that the categorical point of view will emphasize morphisms. But there is a symmetry to be had here. It is easy to check that for any category $\mathcal{C}$, we can define a category $\mathcal{C}^{\mathrm{op}}$ that has $\operatorname{ob}\mathcal{C}^{\mathrm{op}} \coloneqq \operatorname{ob}\mathcal{C}$ but $\operatorname{mor}\mathcal{C}^{\mathrm{op}}$ consists of all morphisms of $\mathcal{C}$ with their domains and codomains interchanged. This new category $\mathcal{C}^{\mathrm{op}}$ is known as the *dual category*. As one may expect, the data of $\mathcal{C}^{\mathrm{op}}$ is exactly the same as the data already carried in $\mathcal{C}$. Nonetheless, the dual category is useful to phrase certain situations where morphisms may become "reversed" in a more efficient way. The morphism $f \in \operatorname{mor}\mathcal{C}$ has its dual in $\operatorname{mor}\mathcal{C}^{\mathrm{op}}$ denoted by f^{op}.

If $m \in \mathbb{N}$ and $f_0, f_1, \ldots, f_m$ are morphisms such that $\operatorname{dom} f_{i+1} = \operatorname{codom} f_i$ for $i \in \{0, 1, \ldots, m-1\}$, we say that the $(m+1)$-tuple $(f_0, f_1, \ldots, f_m)$ is a *composable tuple of morphisms*. By the associativity of composition, any composable tuple of morphisms corresponds to a morphism by just composing all of the components of the tuple.

We will often find ourselves dealing with morphisms that decompose into a composition of many other morphisms. The heart of categorical arguments is that such decompositions, if they exist, are almost never unique. We will often run into the complicated data that arises from a collection of morphisms, decompositions of them, and relations between the constituent morphisms with respect to composition. This data can be expressed efficiently using *commutative diagrams*. To make the notion of a diagram precise, we must first define *functors* which can be interpreted as a natural kind of "morphism between categories".

Definition 1.1.0.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** from $\mathcal{C}$ to $\mathcal{D}$, denoted $F \colon \mathcal{C} \to \mathcal{D}$, associates to every object $X \in \operatorname{ob}\mathcal{C}$ an object $F(X) \in \operatorname{ob}\mathcal{D}$. and to every morphism $f \colon X \to Y$ in $\operatorname{mor}\mathcal{C}$ a morphism $F(f) \colon F(X) \to F(Y)$, such that the following hold.

(1) If $f, g \in \operatorname{mor}\mathcal{C}$ are composable ($\operatorname{dom} f = \operatorname{codom} g$) then so are $F(f)$ and $F(g)$, and $F(f \circ g) = F(f) \circ F(g)$.

(2) For any $X \in \operatorname{ob}\mathcal{C}$, we have that $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$.

A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. ◇

Functors arise all throughout mathematics, and we will encounter some important ones in this book. Functors enable us to take data that is encoded in one category and express it as data in some other category (though sometimes a functor may lose some information; we will soon make precise when does not lose any information).

Functors behave a lot like functions (in fact, they are *class functions*). Hence, we make the following definition.

Definition 1.1.0.3. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a covariant functor.

(1) We define the **object-image** of F to be the subclass

$$\operatorname{im}_{\mathrm{ob}} F \coloneqq \{Y \in \operatorname{ob}\mathcal{D} \colon \exists X \in \operatorname{ob}\mathcal{C} \text{ s.t. } F(X) = Y\}. \tag{1.1.0.1}$$

(2) We define the **morphism-image** of F to be the subclass

$$\operatorname{im}_{\mathrm{mor}} F \coloneqq \{g \in \operatorname{mor}\mathcal{D} \colon \exists f \in \operatorname{mor}\mathcal{C} \text{ s.t. } F(f) = g\}. \tag{1.1.0.2}$$

◇

In general, a functor behaves a lot like a "morphism between categories". We will make this more precise later, but one way in which this is realized is that we can "compose functors" in a natural way, just like we compose functions. We record this below.

Definition 1.1.0.4. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories, and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be covariant functors. The **composition** of F and G is a covariant functor $F \circ G\colon \mathcal{C} \to \mathcal{E}$ defined by the following.

(1) For every $X \in \operatorname{ob}\mathcal{C}$ we have $(F \circ G)(X) \coloneqq F(G(X))$.

(2) For every $f \in \operatorname{mor}\mathcal{C}$ we have $(F \circ G)(f) \coloneqq F(G(f))$. ◇

Note that in Definition 1.1.0.4 we did not prove that our definition of composition actually yields a covariant functor in the sense that it satisfies properties (1) and (2) in Definition 1.1.0.2. However this is easily seen to be the case.

We will now give one more piece of language that succinctly describes a scenario that will arise often. Recall that we noted that morphisms often decompose into a composition of a finite set of morphisms. The same often happens with functors. It is useful to conceptualize this decomposition as existing with respect to the domain of one of the morphisms (or functors) in the decomposition.

Definition 1.1.0.5. Let $\mathcal{C}$ be a category, let $A \in \operatorname{ob}\mathcal{C}$, and let $f\colon X \to Y$ be a morphism of $\mathcal{C}$. We say that f **factors through** A if there exist morphisms $g\colon X \to A$ and $h\colon A \to Y$ such that $f = h \circ g$.

Let $\mathcal{S}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. We say that F **factors through** $\mathcal{S}$ if there exist covariant functors $G\colon \mathcal{C} \to \mathcal{S}$ and $H\colon \mathcal{S} \to \mathcal{D}$ such that $F = H \circ G$. ◇

Armed with these new notions, our first order of business is to define diagrams. This will greatly increase the speed at which we can understand categories.

Definition 1.1.0.6. Let $\mathcal{C}$ be a category. A **diagram** in $\mathcal{C}$ is a covariant functor $\mathscr{D}\colon \mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a small category called the **shape** of the diagram.

A **commutative diagram** in $\mathcal{C}$ is a diagram in $\mathcal{C}$ that factors through a posetal category. ◇

Note that in Definition 1.1.0.6, we draw a distinction between general diagrams and commutative diagrams. Indeed, a diagram need not commute. The abstract definition of a diagram is provided for completeness. The restriction that the shape of a diagram must be small is a technical one we make just to avoid potential issues. For instance, in Theorem 1.1.0.1 we use the smallness of shapes of diagrams in a crucial way.

We will primarily concern ourselves with commutative diagrams in this book. The notion of a commutative diagram is likely already familiar to the reader in several categories (at least on the level of intuition), though it may not be obvious why the abstract definition given in Definition 1.1.0.6 corresponds to this intuitive conception of a commutative diagram. We make the correspondence precise in the following result.

Theorem 1.1.0.1 (Characterization of Commutative Diagrams)**.** *Let $\mathscr{D}\colon \mathcal{J} \to \mathcal{C}$ be a diagram. The following are equivalent.*

(i) *$\mathscr{D}$ is commutative.*

(ii) *For any pair $(f_0, f_1, \ldots, f_m)$ and $(g_0, g_1, \ldots, g_n)$ of composable tuples of morphisms from $\operatorname{mor}\mathcal{J}$, if $\operatorname{dom} f_0 = \operatorname{dom} g_0$ and $\operatorname{codom} f_m = \operatorname{codom} g_n$, then $\mathscr{D}(f_m) \circ \mathscr{D}(f_{m-1}) \circ \cdots \circ \mathscr{D}(f_0) = \mathscr{D}(g_n) \circ \mathscr{D}(g_{n-1}) \circ \cdots \circ \mathscr{D}(g_0)$.*

(iii) *For any pair $(f_0, f_1, \ldots, f_m)$ and $(g_0, g_1, \ldots, g_n)$ of composable tuples of morphisms from $\operatorname{im}_{\operatorname{mor}} \mathscr{D}$, if $\operatorname{dom} f_0 = \operatorname{dom} g_0$ and $\operatorname{codom} f_m = \operatorname{codom} g_n$, then $f_m \circ f_{m-1} \circ \cdots \circ f_0 = g_n \circ g_{n-1} \circ \cdots \circ g_0$.* ◇

Proof. First, suppose $\mathscr{D}$ is commutative. Then, there exist a posetal category $\mathcal{P}$ and covariant functors $F\colon \mathcal{J} \to \mathcal{P}$ and $G\colon \mathcal{P} \to \mathcal{C}$ such that $\mathscr{D} = G \circ F$. Let $(f_0, f_1, \ldots, f_m)$ and $(g_0, g_1, \ldots, g_n)$ be composable tuples of morphisms such that f_i and g_j are in $\operatorname{mor}\mathcal{J}$ for all $i \in \{0, 1, \ldots, m\}$ and $j \in \{0, 1, \ldots, n\}$. It follows that $(F(f_0), F(f_1), \ldots, F(f_m))$ and $(F(g_0), F(g_1), \ldots, F(g_m))$ are composable tuples of morphisms from $\operatorname{mor}\mathcal{P}$. Moreover, we have that

$$\operatorname{dom} F(f_0) = F(\operatorname{dom} f_0) = F(\operatorname{dom} g_0) = \operatorname{dom} F(g_0), \tag{1.1.0.3}$$

$$\operatorname{codom} F(f_m) = F(\operatorname{codom} f_m) = F(\operatorname{codom} g_n) = \operatorname{codom} F(g_n). \tag{1.1.0.4}$$

Therefore, the compositions of the $F(f_i)$ and of the $F(g_j)$ are both elements of the morphism-class $\operatorname{mor}(\operatorname{dom} F(f_0), \operatorname{codom} F(f_m))$. This is a nonempty morphism-class of a posetal category, and thus this morphism-class is a singleton, which implies that the composition of the $F(f_i)$ equals the composition of the $F(g_j)$. Now we can compute

$$\begin{aligned} \mathscr{D}(f_m) \circ \mathscr{D}(f_{m-1}) \circ \cdots \circ \mathscr{D}(f_0) &= G(F(f_m)) \circ G(F(f_{m-1})) \circ \cdots \circ G(F(f_0)) \\ &= G(F(f_m) \circ F(f_{m-1}) \circ \cdots \circ F(f_0)) \\ &= G(F(g_n) \circ F(g_{n-1}) \circ \cdots \circ F(g_0)) \\ &= G(F(g_n)) \circ G(F(g_{n-1})) \circ \cdots \circ G(F(g_0)) \\ &= \mathscr{D}(g_m) \circ \mathscr{D}(g_{m-1}) \circ \cdots \circ \mathscr{D}(g_0). \end{aligned} \tag{1.1.0.5}$$

Hence, (i) implies (ii).

Conversely, suppose that every pair of composable tuples of morphisms from $\operatorname{mor}\mathcal{J}$ satisfies the property given in (ii). Let us abbreviate this property as the *commutativity property*. Let S be the class of all composable tuples of morphisms from $\operatorname{mor}\mathcal{J}$. Since $\mathcal{J}$ is a small category, S is actually a set. Define the binary relation $\sim$ on S by $(p_0, p_1, \ldots, p_m) \sim (q_0, q_1, \ldots, q_n)$ if and only if $\operatorname{dom} p_0 = \operatorname{dom} q_0$ and $\operatorname{codom} p_m = \operatorname{codom} q_n$. We claim that $\sim$ is an equivalence relation on S. Indeed, $\sim$ is easily seen to inherit reflexivity, symmetry, and transitivity from $=$.

Hence, we may consider the quotient set $S/\sim$, denoting the equivalence class of any $p \in S$ by $\overline{p}$. We define a category $\mathcal{P}$ as follows. Define

$$\operatorname{ob}\mathcal{P} := \{\operatorname{dom} f_0 \colon (f_0, f_1, \ldots, f_m) \in S\} \cup \{\operatorname{codom} f_m \colon (f_0, f_1, \ldots, f_m) \in S\} = \operatorname{ob}\mathcal{J}, \tag{1.1.0.6}$$

$$\operatorname{mor}\mathcal{P} := S/\sim . \tag{1.1.0.7}$$

The last equality in Equation 1.1.0.6 follows from the fact that every 1-tuple of morphisms from $\operatorname{mor}\mathcal{J}$ is an element of S.

For every $f = \overline{(f_0, f_1, \ldots, f_m)} \in \operatorname{mor}\mathcal{P}$ we define $\operatorname{dom} f := \operatorname{dom} f_0$ and $\operatorname{codom} f := \operatorname{codom} f_m$. Note that these are well-defined by definition of the equivalence relation $\sim$. If $g = \overline{(g_0, g_1, \ldots, g_n)} \in \operatorname{mor}\mathcal{P}$ is an equivalence class with $\operatorname{codom} g = \operatorname{dom} f$, then we may define the composition $f \circ g$ to be the equivalence class $\overline{(g_0, g_1, \ldots, g_n, f_0, f_1, \ldots, f_m)}$. This composition operation is well-defined by our definition of the equivalence relation $\sim$. Composition is also clearly associative. Finally, for any object $X \in \operatorname{ob}\mathcal{P}$, it is clear that $\overline{(\mathbf{1}_X)}$ behaves as an identity morphism. Hence, $\mathcal{P}$ is indeed a category.

For $X, Y \in \operatorname{ob}\mathcal{P}$, suppose $\operatorname{mor}_{\mathcal{P}}(X, Y)$ is nonempty. Pick $f, g \in \operatorname{mor}_{\mathcal{P}}(X, Y)$. Then we may write $f = \overline{(f_0, f_1, \ldots, f_m)}$ and $g = \overline{(g_0, g_1, \ldots, g_n)}$ where the f_i and g_j form composable tupes of morphisms in $\operatorname{mor}\mathcal{J}$. Since $\operatorname{dom} f_0 = X = \operatorname{dom} g_0$ and $\operatorname{codom} f_m = Y = \operatorname{codom} g_n$, we must have that $f = g$. Hence, $\mathcal{P}$ is a posetal category.

Now, we may define the covariant functor $\Pi \colon \mathcal{J} \to \mathcal{P}$ given by $\Pi(X) := X$ for all $X \in \operatorname{ob}\mathcal{J}$ and $\Pi(f) := \overline{(f)}$ for all $f \in \operatorname{mor}\mathcal{J}$. Define a covariant functor $G \colon \mathcal{P} \to \mathcal{C}$ by setting $G(X) := \mathscr{D}(X)$ for all $X \in \operatorname{ob}\mathcal{P}$ and for every $\overline{(f_0, f_1, \ldots, f_m)} \in \operatorname{mor}\mathcal{P}$ define $G(\overline{(f_0, f_1, \ldots, f_m)}) := \mathscr{D}(f_m \circ f_{m-1} \circ \cdots \circ f_0) = \mathscr{D}(f_m) \circ \mathscr{D}(f_{m-1}) \circ \cdots \circ \mathscr{D}(f_0)$. This is well-defined by the definition of $\sim$ and because every pair of elements of S enjoys the commutativity property by assumption. Now it is immediate that $\mathscr{D} = G \circ \Pi$, so $\mathscr{D}$ factors through $\mathcal{P}$. Hence, (ii) implies (i).

The equivalence of (ii) and (iii) is simple to see. Suppose (ii) holds and let $(f_0, f_1, \ldots, f_m)$ and $(g_0, g_1, \ldots, g_n)$ are composable tuples of morphisms from $\operatorname{im}_{\mathrm{mor}}\mathscr{D}$ with $\operatorname{dom} f_0 = \operatorname{dom} g_0$ and $\operatorname{codom} f_m = \operatorname{codom} g_n$. Since all of the morphisms are from $\operatorname{im}_{\mathrm{mor}}\mathscr{D}$, we may write $f_i = \mathscr{D}(\tilde{f}_i)$ and $g_j = \mathscr{D}(\tilde{g}_j)$ for all $i \in \{0, 1 \ldots, m\}$ and $j \in \{0, 1, \ldots n\}$. By the functoriality of $\mathscr{D}$, we have that $(\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_m)$ and $(\tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_n)$ are composable tuples of morphisms from $\operatorname{mor}\mathcal{J}$ with $\operatorname{dom} \tilde{f}_0 = \operatorname{dom} \tilde{g}_0$ and $\operatorname{codom} \tilde{f}_m = \operatorname{codom} \tilde{g}_n$. Hence by (ii), $f_m \circ f_{m-1} \dots f_0 = g_n \circ g_{n-1} \circ \dots \circ g_0$ as desired.

Conversely, suppose (iii) holds and let $(f_0, f_1, \dots, f_m)$ and $(g_0, g_1, \dots, g_n)$ be composable tuples of morphisms from $\operatorname{mor} \mathcal{J}$. By functoriality, $(\mathscr{D}(f_0), \mathscr{D}(f_1), \dots, \mathscr{D}(f_m))$ and $(\mathscr{D}(g_0), \mathscr{D}(g_1), \dots, \mathscr{D}(g_n))$ are composable tuples of morphisms from $\operatorname{im}_{\mathrm{mor}} \mathscr{D}$ with $\operatorname{dom} \mathscr{D}(f_0) = \operatorname{dom} \mathscr{D}(g_0)$ and $\operatorname{codom} \mathscr{D}(f_m) = \operatorname{codom} \mathscr{D}(g_n)$. So by (iii), we have that $\mathscr{D}(f_m) \circ \mathscr{D}(f_{m-1}) \circ \dots \circ \mathscr{D}(f_0) = \mathscr{D}(g_n) \circ \mathscr{D}(g_{n-1}) \circ \dots \mathscr{D}(g_0)$, which establishes (iii). □

Note that in the proof of Theorem 1.1.0.1, we used the fact that under our definition, the shape of a diagram must be small. This ensures that S is a set and thus any equivalence class of $\sim$ must be a set. Without the smallness of the shape of the diagram, the equivalence classes may be proper classes, which would mean $S/\sim$ cannot exist in NBG set theory as there is no class in that theory which contains a proper class as an element.

Note that every diagram in a posetal category is commutative. Indeed, if $\mathscr{D} \colon \mathcal{J} \to \mathcal{P}$ is a diagram in a posetal category then $\mathscr{D}$ factors through $\mathcal{P}$ itself via the identity functor $\mathcal{P} \to \mathcal{P}$. So every commutative diagram is essentially a diagram with small posetal shape. The proof of Theorem 1.1.0.1 shows us that the posetal category that a commutative diagram factors through holds the "essential data" of the diagram, i.e. the data that remains after identifying all equal composites in the shape of the diagram.

Condition (iii) in Theorem 1.1.0.1 is what we intuitively think of when we think of a commutative diagram. To see this, first we note that if $\mathscr{D} \colon \mathcal{J} \to \mathcal{C}$ is a diagram, then we may interpret $\mathscr{D}$ as a *quiver* which is a directed graph that is permitted to have multiple directed edges between any two vertices. We do this by considering the quiver whose collection of vertices is $\operatorname{im}_{\mathrm{ob}} \mathscr{D}$ and whose collection of directed edges is just $\operatorname{im}_{\mathrm{mor}} \mathscr{D}$. We direct the edges so that each directed edge is an arrow traveling from the domain to the codomain of the corresponding morphism.

As a prototypical example, one may identify any small category $\mathcal{C}$ with a diagram over itself, namely the identity functor $\mathcal{C} \to \mathcal{C}$. For instance, if the identity functor of a small category $\mathcal{C}$ has the associated quiver

$$\begin{tikzcd} & Y \arrow[loop above, "\mathbf{1}_Y"] \arrow[dr, "g"] & \\ X \arrow[loop left, "\mathbf{1}_X"] \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & Z \arrow[loop right, "\mathbf{1}_Z"] \end{tikzcd} \tag{1.1.0.8}$$

then we can quickly understand that $\mathcal{C}$ is a posetal category with $\operatorname{ob} \mathcal{C} = \{X, Y, Z\}$ and non-identity morphisms $f \colon X \to Y$, $g \colon Y \to Z$, and $h \colon X \to Z$ where $h = g \circ f$. This is a powerful way of thinking of a category (at least finite categories whose quivers can be completely drawn).

We will often identify a diagram with its quiver and just refer to the quiver as the diagram itself. We will draw a quiver completely in black if we know that it is the quiver of commutative diagram and we wish to emphasize that that particular diagram commutes (is commutative). Otherwise, we will use red for the vertices and blue for the edges of the quiver. For instance, in Diagram 1.1.0.8 above, we used color, but since $\mathcal{C}$ is posetal and every diagram in a posetal category commutes, we could have drawn the entire diagram in black. We will also often omit the identity morphisms in diagrams.

Now, we can see that Theorem 1.1.0.1 says that a diagram is commutative if and only if the associated quiver has the property that for any pair of vertices A and B of the quiver, any path taken by traveling along the directed edges from A to B corresponds to the same morphism after one composes all directed edges along the path. Thus, we have recovered the intuitive notion of a commutative diagram from the abstract Definition 1.1.0.6.

1.1.1 Examples

We now give several examples of categories and functors to illustrate that these constructions arise naturally throughout all of mathematics.

Example 1.1.1.1. For any category $\mathcal{C}$, we can often find another category $\mathcal{D}$ "living inside" the category $\mathcal{C}$ in a natural way. For a category $\mathcal{C}$, a *subcategory* of $\mathcal{C}$ consists of the data of a subcollection $\operatorname{ob}\mathcal{D}$ of objects from $\operatorname{ob}\mathcal{C}$ and a subcollection $\operatorname{mor}\mathcal{D}$ of morphisms from $\operatorname{mor}\mathcal{C}$ such that the following conditions hold.

(1) For every $X \in \operatorname{ob}\mathcal{D}$ we have $\mathbf{1}_X \in \operatorname{mor}\mathcal{D}$.

(2) For every morphism $f \in \operatorname{mor}\mathcal{D}$ we have $\operatorname{dom} f \in \operatorname{ob}\mathcal{D}$ and $\operatorname{codom} f \in \operatorname{ob}\mathcal{D}$.

(3) For every morphism $f, g \in \operatorname{mor}\mathcal{D}$ we have if $f \circ g$ is defined in $\operatorname{mor}\mathcal{C}$ then $f \circ g \in \operatorname{mor}\mathcal{D}$.

Of course, a subcategory of a category is itself a category. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a *full subcategory* of $\mathcal{C}$ if for every pair of objects $X, Y \in \mathcal{D}$ we have $\operatorname{mor}_{\mathcal{D}}(X,Y) = \operatorname{mor}_{\mathcal{C}}(X,Y)$. Of course, there is also a natural *inclusion functor* $\mathcal{D} \to \mathcal{C}$ that sends objects and morphisms to themselves. $\diamond$

Example 1.1.1.2. We define the category Set to be the category whose objects are sets and whose morphisms are functions between sets, with domains, codomains, compositions, and identity functions being defined in the usual set-theoretic way. Note that in this category, the empty set is an initial object, singletons are final objects, and there are no zero objects. This category forms the intuitive basis for the definition of categories. If $\mathcal{D}$ is a subcategory of $\mathcal{C}$ which is itself a subcategory of Set, the inclusion functor $\mathcal{D} \to \mathcal{C}$ is often referred to as the *forgetful functor*, since it "forgets" any structure inherent to $\mathcal{D}$.

Many common categories can be realized as subcategories of Set. We list some frequently encountered ones below.

(1) Grp is the category of groups and group homomorphisms. The trivial group is a zero object in this category.

(2) Ab is the category of Abelian groups and group homomorphisms. This is a full subcategory of Grp. This is also the prototypical example of an *Abelian category* which we will define in the next chapter.

There exists a covariant functor $\mathscr{A}\colon \mathsf{Grp} \to \mathsf{Ab}$. This functor sends every group G to its *Abelianization* $\mathscr{A}(G) \coloneqq G/[G,G]$. Any group homomorphism $f\colon G \to H$ also induces a group homomorphism $\mathscr{A}(f)\colon \mathscr{A}(G) \to \mathscr{A}(H)$. Let $\pi_G\colon G \to G/[G,G]$ and $\pi_H\colon H \to H/[H,H]$ be quotient maps. For every element $\pi_G(g) \in G/[G,G]$, we define $[\mathscr{A}(f)](\pi_G(g)) \coloneqq \pi_H(f(g))$. Indeed this is well-defined, since for any $x \in [G,G]$, we will have that x can be written as a product of commutators in G, and since f is a group homomorphism, $f(x)$ will thus be a product of commutators in H, hence $f(x) \in [H,H]$. This functor is of importance in topology.

(3) Given a topological space X and a category $\mathcal{C}$, we define $\mathsf{PSh}_{\mathcal{C}}(X)$ to be the category of presheaves on X, with the presheaves taking values in $\mathcal{C}$. We define $\mathsf{Sh}_{\mathcal{C}}(X)$ to be the category of sheaves on X, with the sheaves taking values in $\mathcal{C}$. Note that $\mathsf{Sh}_{\mathcal{C}}(X)$ is a full subcategory of $\mathsf{PSh}_{\mathcal{C}}(X)$. Sheafification is a functor $\mathsf{PSh}_{\mathcal{C}}(X) \to \mathsf{Sh}_{\mathcal{C}}(X)$.

(4) Ring is the category of rings and ring homomorphisms. Here, we do not insist that the rings need to be commutative. The ring of integers $\mathbb{Z}$ is an initial object and the zero ring is a final object. We define CRing to be the category of commutative rings and ring homomorphisms. CRing is a full subcategory of Ring.

Sch is the category of schemes and morphisms between schemes. This is the category that is primarily studied in algebraic geometry, though it is not quite a subcategory of Set since morphisms of schemes are morphisms of locally ringed spaces, which carry more data than just a single function between the underlying sets of the schemes. We define ASch to be the category of affine schemes and morphisms between them. ASch is a full subcategory of Sch. Moreover, there exists a contravariant functor $\operatorname{Spec}\colon \mathsf{CRing}^{\mathrm{op}} \to \mathsf{ASch}$ which takes every commutative ring to its spectrum and takes morphisms between commutative rings to the induced morphisms between their spectra. This furnishes an *equivalence* between $\mathsf{CRing}^{\mathrm{op}}$ and ASch in a sense which we will make precise in the next section. Hence, we may think of ASch as being essentially the same as the dual category $\mathsf{CRing}^{\mathrm{op}}$.

(5) Given a fixed ring R, we define $R\text{-}\mathsf{Mod}$ to be the category of left R-modules and R-module homomorphisms.

(6) If k is a field, then k-modules are just k-vector spaces and k-module homomorphisms are just k-linear homomorphisms. Therefore, the category $k\text{-}\mathsf{Mod}$ is often denoted by Vect_k. A full subcategory of this is $\mathsf{FinDimVect}_k$, which is the category of finite-dimensional k-vector spaces and k-linear homomorphisms between them. When $k = \mathbb{R}$, a relevant category arises when we add a little more structure to obtain Ban, which is the category of real Banach spaces and continuous linear maps between them.

(7) Relevant to analysis is the category cHaus which is the category of compact Hausdorff spaces and continuous maps between them.

(8) Man^∞ is the category of smooth (infinitely differentiable) manifolds and smooth functions. We may modify this category to construct the category Man_*^∞, which is the category of *pointed smooth manifolds*. That is, objects of Man_*^∞ are ordered pairs (M, p) where M is a smooth manifold and p is a point in M. A morphism $(M, p) \to (N, q)$ of Man_*^∞ is the data of a smooth map $f\colon M \to N$ such that $f(p) = q$. Hence, Man_*^∞ is a category of smooth manifolds that keeps track of "basepoints".

One of the first and most important constructions in differential geometry is a covariant functor $\mathsf{Man}_*^\infty \to \mathsf{Vect}_{\mathbb{R}}$. To any pointed smooth manifold (M, p) we can associate the real vector space $T_p(M)$. Moreover, any smooth map $f\colon M \to N$ between two smooth manifolds induces a $\mathbb{R}$-linear homomorphism $\mathrm{d}f\colon T_p(M) \to T_{f(p)}(N)$. This induced map is the *differential* of the smooth map f. The differential is functorial, so tangent spaces along with the differentials of smooth maps give a covariant functor $\mathsf{Man}_*^\infty \to \mathsf{Vect}_{\mathbb{R}}$. We refer to this functor as the *linearization functor*. The importance of the linearization functor to differential geometry cannot be understated. It essentially allows problems in geometry to be translated into problems in linear algebra.

(9) Top is the category of topological spaces and continuous functions. Similar to the smooth category Man^∞, we may modify Top to obtain a category of pointed topological spaces Top_*.

In some sense, this book is a study of the categories Top and Top_*. This will be accomplished in part by constructing various functors of the form $\mathsf{Top}_* \to \mathsf{Grp}$, $\mathsf{Top} \to \mathsf{Ab}$, and $\mathsf{Top} \to \mathsf{Ring}$ (though we will also construct other categories related to Top and study them as well). However, these functors will be useful for a different reason than the linearization functor is in differential geometry. While the linearization functor allows us to turn geometry problems into linear algebra problems, it is a rather weak invariant. This is because up to isomorphism, there is only one vector space of dimension d for any positive integer d. Hence, the linearization functor is incapable of distinguishing between smooth manifolds of the same dimension. Said another way, the linearization functor is only as strong of an invariant as dimension is—which is not very strong.

On the other hand, the functors we will construct out of Top and Top_* will land in very rich categories such as Grp and Ring. We will give a general categorical definition of isomorphism in the next section, but the reader may recognize that there are many isomorphism classes in these categories. So our functors will be powerful invariants of topological spaces, giving us a way of distinguishing between many topological spaces. On the other hand, problems in group theory and ring theory are typically much more difficult than problems in linear algebra, so our functors will not serve as well as the linearization functor as a tool for *solving* problems in geometry and topology by viewing them as problems in algebra.

Categories that cannot be identified with a subcategory of Set are quite rare. The failure of a category to be identified with a subcategory of Set is a always contigent upon the category being "too large" in the sense that its object class or morphism class are not sets but proper classes. In the next section, we will give a general definition of *concrete categories*, which are categories that come equipped with an "identification" with a subcategory of Set. Much later on, we will find an example of a category that is not concretizable. $\diamond$

Example 1.1.1.3. Given a ring R, we define Mat_R to be the category whose objects are nonnegative integers and whose morphisms $n \to m$ for $n, m > 0$ are $m \times n$ matrices whose entries are in R and 0 is a zero object. Composition of morphisms correspond to matrix multiplication. This example of a category may seem a bit contrived but it actually represents the concrete manner in which we think of finite-dimensional linear algebra in a way we will make precise in Example 1.2.1.2. $\diamond$

Example 1.1.1.4. A *monoid* is an algebraic structure with a binary operation which is associative and admits an identity element. Such a structure carries the same data as a category with a single object, where the morphisms of the category represent the elements of the monoid, the binary operation is composition, and the identity element is the identity morphism of the single object in the category.

Note that all groups are monoids. In particular, a group is a monoid such that all morphisms have an inverse (i.e. every morphism is an *isomorphism* in the sense of the upcoming Definition 1.2.0.3). Let G be a group and let $\mathcal{G}$ be the category representing the group G. Let $\mathcal{C}$ be an arbitrary subcategory of Set. A natural question is: what data does a covariant functor $F\colon \mathcal{G} \to \mathcal{C}$ hold?

First note that $\operatorname{ob}\mathcal{G}$ consists of a single object X and thus $\operatorname{im}_{\text{ob}} F$ consists of a single object $F(X)$. Since $\mathcal{C}$ is a subcategory of Set, the object $F(X)$ is a set. Every element $g \in G$ can be interpreted as a morphism $\tilde{g} \in \operatorname{mor}\mathcal{G}$. This morphism in turn determines a unique function $F(\tilde{g})\colon F(X) \to F(X)$. We can see that for $g, h \in G$, we will have by functoriality $F(\tilde{g} \circ \tilde{h}) = F(\tilde{g}) \circ F(\tilde{h})$. Moreover, if e is the identity element of G, then $\tilde{e} = \mathbf{1}_X$ so $F(\tilde{e}) = \mathbf{1}_{F(X)}$. It follows that the covariant functor F actually holds the data of a left group action of G on the set $F(X)$. It is easy to check that the converse is also true: a left group action induces a covariant functor. Similarly, a right group action of G can be interpreted as a contravariant functor from $\mathcal{G}$ to $\mathcal{C}$.

Another perspective we may take is when $\mathcal{C}$ is itself a category representing a group H. Then, the functoriality of F shows us that F carries the data of a group homomorphism $G \to H$. $\diamond$

Example 1.1.1.5. Cat is the is the category of small categories and covariant functors. It is essential that we only take small categories since otherwise our collection of objects would fail to be a class. This is an example of a *large* (not small) category. $\diamond$

Once we discuss more terminology in the next section, we will be equipped to give more examples of categories.

Example 1.1.1.6. Let $\mathcal{C}$ be a locally small category and let $X \in \operatorname{ob}\mathcal{C}$. The *covariant hom-functor* is a covariant functor $\operatorname{Hom}(X, -)\colon \mathcal{C} \to \mathsf{Set}$ that maps every object $Y \in \operatorname{ob}\mathcal{C}$ to the morphism-class $\operatorname{mor}(X, Y)$ and every morphism $f\colon Y \to Z$ to the morphism $\operatorname{Hom}(X, -)(f)\colon \operatorname{mor}(X, Y) \to \operatorname{mor}(X, Z)$ given by $[\operatorname{Hom}(X, -)(f)](\varphi) \coloneqq f \circ \varphi$ for every $\varphi \in \operatorname{mor}(X, Y)$. The reason for the notation Hom will be made more clear in the next chapter.

In a similar vein, we may define the *contravariant hom-functor* $\operatorname{Hom}(-, X)\colon \mathcal{C}^{\text{op}} \to \mathsf{Set}$ that maps each object $Y \in \operatorname{ob}\mathcal{C}$ to the morphism-class $\operatorname{mor}(Y, X)$ and every morphism $f\colon Y \to Z$ from $\operatorname{mor}\mathcal{C}^{\text{op}}$ to the morphism $\operatorname{Hom}(-, X)(f)\colon \operatorname{mor}(Y, X) \to \operatorname{mor}(Z, X)$ that acts via $[\operatorname{Hom}(-, X)(f)](\varphi) \coloneqq \varphi \circ \tilde{f}$ for every $\varphi \in \operatorname{mor}(Y, X)$, where $\tilde{f}\colon Z \to Y$ is the unique morphism from $\operatorname{mor}\mathcal{C}$ such that $\tilde{f}^{\text{op}} = f$. $\diamond$

Example 1.1.1.7. Let k be a field. There is a contravariant functor Φ^* from $\mathsf{Vect}_k^{\text{op}}$ to Vect_k called the *dualization functor*. This functor sends every vector space to its algebraic dual and associates to every linear transformation of vector spaces $f\colon V \to W$ the dual map $f^*\colon W^* \to V^*$ defined by $f^*(\varphi) \coloneqq \varphi \circ f$. Note that the data of the dual functor Φ^* is precisely the same as that of the contravariant hom-functor $\operatorname{Hom}(-, k)$ where k is treated as a one-dimensional vector space over itself.

We may also define a covariant functor Φ^{**} called the *double dualization functor*. It behaves by sending every vector space to its algebraic double dual and every linear transformation of vector spaces $f\colon V \to W$ to the double dual map $f^{**}\colon V^{**} \to W^{**}$ defined in the following way: for every $\varphi \in V^{**}$, we define $f^{**}(\varphi) \in W^{**}$ to be the map $W^* \to k$ given by $[f^{**}(\varphi)](\psi) \coloneqq \varphi(\psi \circ f)$ for every $\psi \in W^*$. $\diamond$

Example 1.1.1.8. Commutative diagrams are ubiquitous throughout mathematics. For example, in Grp we have a commutative diagram which arises from the first isomorphism theorem quoted below.

> Let G and H be groups and let $\varphi\colon G \to H$ be a group homomorphism. Then, φ factors uniquely through the quotient map $\pi\colon G \to G/\ker\varphi$. Moreover, the map $\tilde{\varphi}\colon G/\ker\varphi \to H$ in this factorization is injective.

We may summarize the content of the first isomorphism theorem with the following commutative diagram.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"', two heads] & H \\ G/\ker\varphi \arrow[ur, hook, dashed, "\tilde{\varphi}"'] & \end{tikzcd} \tag{1.1.1.1}$$

In this diagram we adopt a few more conventions. The double-headed arrow $\pi\colon G \twoheadrightarrow G/\ker\varphi$ indicates that the quotient map π is surjective, and the hook-ended arrow $\tilde{\varphi}\colon G/\ker\varphi \hookrightarrow H$ indicates that the induced map $\tilde{\varphi}$ is injective. This arrow is also dotted to indicate that the content of the theorem is the existence of this map, i.e. that the map is *induced* from the given maps φ and π. $\diamond$

Example 1.1.1.9. Let $\mathcal{C}$ be a category. Consider the category of sheaves $\mathsf{Sh}_{\mathcal{C}}(X)$ taking values in $\mathcal{C}$. We can succinctly describe the data of morphisms in $\mathsf{Sh}_{\mathcal{C}}(X)$ with commutative diagrams in $\mathcal{C}$.

Let $\mathscr{F}$ and $\mathscr{G}$ be objects of $\mathsf{Sh}_{\mathcal{C}}(X)$. For any inclusion of open subsets $U \subseteq V \subseteq X$, let $\rho^{\mathscr{F}}_{U,V}\colon \mathscr{F}(V) \to \mathscr{F}(U)$ be the corresponding restriction morphism of $\mathscr{F}$ and likewise $\rho^{\mathscr{G}}_{U,V}\colon \mathscr{G}(V) \to \mathscr{G}(U)$ for $\mathscr{G}$. Let $\Phi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then Φ is the data of a collection of morphisms $\Phi_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ indexed by the open subsets $U \subseteq X$ such that for every inclusion of open subsets $U \subseteq V \subseteq X$, the following diagram in $\mathcal{C}$ commutes.

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\Phi_V"] \arrow[d, "\rho^{\mathscr{F}}_{U,V}"'] & \mathscr{G}(V) \arrow[d, "\rho^{\mathscr{G}}_{U,V}"] \\ \mathscr{F}(U) \arrow[r, "\Phi_U"'] & \mathscr{G}(U) \end{tikzcd} \tag{1.1.1.2}$$

Commutative diagrams with the same shape as in Diagram 1.1.1.2 are said to be *commuting squares*. $\diamond$

Factorization diagrams as in Example 1.1.1.8 and commuting squares as in Example 1.1.1.9 are the most common types of commutative diagrams which we will encounter. We may also use commuting squares to give an example of a category whose objects are not sets of any kind and whose morphisms are therefore not functions.

Example 1.1.1.10. Let $\mathcal{C}$ be a category. We define the category $\mathsf{Mor}(\mathcal{C})$ to be the category whose objects are the morphisms of $\mathcal{C}$ and whose morphisms are commuting squares. That is, for $f, g \in \operatorname{ob}\mathsf{Mor}(\mathcal{C}) = \operatorname{mor}\mathcal{C}$, a morphism $\Phi\colon f \to g$ is the data of two morphisms $\Phi_1\colon \operatorname{dom} f \to \operatorname{dom} g$ and $\Phi_2\colon \operatorname{codom} f \to \operatorname{codom} g$ such that the following diagram commutes.

$$\begin{tikzcd} \operatorname{dom} f \arrow[r, "\Phi_1"] \arrow[d, "f"'] & \operatorname{dom} g \arrow[d, "g"] \\ \operatorname{codom} f \arrow[r, "\Phi_2"'] & \operatorname{codom} g \end{tikzcd} \tag{1.1.1.3}$$

Note that for any $f \in \operatorname{ob}\mathsf{Mor}(\mathcal{C})$ the identity morphism $\mathbf{1}_f\colon f \to f$ is the morphism arising from the commuting square involving the two morphisms $(\mathbf{1}_f)_1 = \mathbf{1}_{\operatorname{dom} f}$ and $(\mathbf{1}_f)_2 = \mathbf{1}_{\operatorname{codom} f}$. $\diamond$

Example 1.1.1.11. Let $\mathcal{C}$ be a subcategory of Set. Suppose the following diagram commutes in $\mathcal{C}$.

$$\begin{tikzcd} & A \arrow[dl, "f"'] & \\ B \arrow[rr, "g"'] & & C \arrow[ul, "h"'] \end{tikzcd} \tag{1.1.1.4}$$

Apart from factorization diagrams such as in Diagram 1.1.1.1, "cycle diagrams" where the morphisms make a complete circuit around the diagram as in Diagram 1.1.1.4 are the only other possible "triangular" diagrams. We study this diagram for the subcategory $\mathcal{C}$ of Set. It will be useful to not suppress the identity morphisms in this commutative diagram. The complete diagram is as follows.

$$\begin{tikzcd}
 & A \arrow[loop above, "\mathbf{1}_A"] \arrow[dl, "f"'] & \\
B \arrow[loop left, "\mathbf{1}_B"] \arrow[rr, "g"'] & & C \arrow[loop right, "\mathbf{1}_C"] \arrow[ul, "h"']
\end{tikzcd} \tag{1.1.1.5}$$

Suppose $a, b \in A$ are chosen so that $f(a) = f(b)$. Then, by the commutativity of Diagram 1.1.1.5, we have that

$$a = \mathbf{1}_A(a) = h(g(f(a))) = h(g(f(b))) = \mathbf{1}_A(b) = b. \tag{1.1.1.6}$$

Therefore, f is injective. Similar arguments show that g and h are injective as well.

Now, pick $a \in A$. We have that $f(a) = f(h(g(f(a))))$. By the injectivity of f, we have that $a = h(g(f(a)))$. This means that a is in the image of h and thus h is surjective. Similar arguments show that f and g are surjective. Hence, all morphisms in this diagram are bijections. ◇

1.2 Naturality and Equivalence

In Section 1.1.1 we gave many examples of subcategories of Set. In this section, we make a more general notion of *concrete categories* and we also make note of the fundamental notions of natural transformations, isomorphisms, and equivalences which will arise in the discussion.

Definition 1.2.0.1. A functor is said to be **faithful** if it is injective on morphism-classes. A functor is said to be **full** if it is surjective on morphism-classes. ◇

Definition 1.2.0.2. A category $\mathcal{C}$ equipped with a faithful functor $\mathcal{C} \to \mathsf{Set}$ is said to be a **concrete category**. A category $\mathcal{D}$ that admits a faithful functor $\mathcal{D} \to \mathsf{Set}$ is said to be **concretizable**. ◇

The subcategories of Set which we discussed in Section 1.1.1 are naturally concretizable as they become concrete when equipped with their forgetful functors. In fact, a converse is also true: any concrete category is *isomorphic* to a subcategory of Set. We make precise what we mean by "isomorphism". There is a notion of an isomorphism within a category and also a notion of isomorphism between two categories. We start with the former.

Definition 1.2.0.3. Let $\mathcal{C}$ be a category and let $X, Y \in \operatorname{ob}\mathcal{C}$ be objects and $f \in \operatorname{mor}(X, Y)$ a morphism. We say that f is an **isomorphism** if there exists $f^{-1} \in \operatorname{mor}(Y, X)$ such that $f^{-1} \circ f = \mathbf{1}_X$ and $f \circ f^{-1} = \mathbf{1}_Y$. We denote X and Y being isomorphic by writing $X \cong Y$. ◇

This notion of isomorphism conicides with what is usually seen in the familiar subcategories of Set such as the ones given in Example 1.1.1.2. We can say something interesting about initial and final objects with this notion.

Proposition 1.2.0.1. *Initial and final objects are unique up to unique isomorphism.*

Proof. Let X and X' be initial objects. Then there exist morphisms $f\colon X \to X'$ and $g\colon X' \to X$. The compositions $f \circ g\colon X' \to X'$ and $g \circ f\colon X \to X$ must be the identity morphisms since $\operatorname{mor}(X, X)$ and $\operatorname{mor}(X', X')$ are singletons containing the identity morphism. These isomorphisms are also unique since $\operatorname{mor}(X, X')$ and $\operatorname{mor}(X', X)$ are singletons. The same argument applies to final objects. □

Note that in Example 1.1.1.5 we noted small categories and covariant functors form a category Cat. Intepreting Definition 1.2.0.3 in Cat, we arrive a notion of an *isomorphism of small categories*. That is, we say that two small categories are isomorphic if there exist functors going both ways such that both of the composition functors are the identity functors. We can generalize this notion to all categories.

Definition 1.2.0.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an **isomorphism** between the categories $\mathcal{C}$ and $\mathcal{D}$ is there exists a functor $F^{-1}\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ F^{-1}$ and $F^{-1} \circ F$ are the identity functors. We denote $\mathcal{C}$ and $\mathcal{D}$ being isomorphic by writing $\mathcal{C} \cong \mathcal{D}$. ◇

An important fact is that if $X \cong \tilde{X}$ and $Y \cong \tilde{Y}$, then we have that $\operatorname{mor}(X, Y)$ is "isomorphic" to $\operatorname{mor}(\tilde{X}, \tilde{Y})$ is a certain natural sense. More precisely, we have the following.

Proposition 1.2.0.2. *Let $\mathcal{C}$ be a category and let $X, Y \in \operatorname{ob}\mathcal{C}$. Let $\varphi\colon X \to \tilde{X}$ and $\psi\colon Y \to \tilde{Y}$ be isomorphisms. Then every morphism $f\colon X \to Y$ induces a unique morphism $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ such that the following diagram commutes.*

$$\begin{tikzcd}
X \arrow[r, bend left, "\varphi"] \arrow[d, "f"'] & \tilde{X} \arrow[l, bend left, "\varphi^{-1}"] \arrow[d, dashed, "\tilde{f}"] \\
Y \arrow[r, bend right, "\psi"'] & \tilde{Y} \arrow[l, bend right, "\psi^{-1}"']
\end{tikzcd} \tag{1.2.0.1}$$

◇

Proof. Define $\tilde{f} \coloneqq \psi \circ f \circ \varphi^{-1}$. Note that this identity is forced by Diagram 1.2.0.1, so uniqueness is immediate. It remains to check that $\psi^{-1} \circ \tilde{f} = f \circ \varphi^{-1}$, $\tilde{f} \circ \varphi = \psi \circ f$, and $f = \psi^{-1} \circ \tilde{f} \circ \varphi$. But all of these identities are immediate consequences of algebraically manipulating the defining identity $\tilde{f} = \psi \circ f \circ \varphi^{-1}$ by composing morphisms with their inverses as necessary. □

Using this notion of isomorphism of categories, we can state the correspondence between concrete categories and subcategories of Set.

Theorem 1.2.0.1. *A category is concretizable if and only if it is isomorphic to a subcategory of* Set. ◇

Proof. In one direction, suppose $\mathcal{C}$ is isomorphic to a subcategory of Set. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an isomorphism where $\mathcal{D}$ is a subcategory of Set. Let $G\colon \mathcal{D} \to \mathsf{Set}$ be the forgetful functor. The composition $G \circ F$ is a functor $\mathcal{C} \to \mathsf{Set}$. We will show that this functor is faithful.

Let $f, g \in \operatorname{mor}\mathcal{C}$ be morphisms such that $(G \circ F)(f) = (G \circ F)(g)$. Since the forgetful functor is faithful, it follows that $F(f) = F(g)$. Since F is an isomorphism, there exists a functor $F^{-1}\colon \mathcal{D} \to \mathcal{C}$ such that $F^{-1} \circ F$ and $F \circ F^{-1}$ are the identity functors. Applying the functor F^{-1} to both sides of the identity $F(f) = F(g)$ yields $f = g$. This establishes that $G \circ F$ is faithful as claimed.

In the other direction, suppose $\mathcal{C}$ is concretizable. Let $F\colon \mathcal{C} \to \mathsf{Set}$ be a faithful functor. We can define a functor $\tilde{F}\colon \mathcal{C} \to \mathsf{Set}$ by $\tilde{F}(X) \coloneqq F(X) \cup \{(X, F(X))\}$ for every $X \in \operatorname{ob}\mathcal{C}$ and $\tilde{F}(f) \coloneqq f'$ for each $f\colon X \to Y$ in $\operatorname{mor}\mathcal{C}$, where $f'\colon \tilde{F}(X) \to \tilde{F}(Y)$ is given by

$$f'(x) \coloneqq \begin{cases} [F(f)](x) & x \in F(X) \\ (Y, F(Y)) & x = (X, F(X)). \end{cases} \tag{1.2.0.2}$$

First, note that by construction, $\tilde{F}$ is faithful because F is. Moreover, $\tilde{F}$ is clearly injective on objects, since we explicitly record the input object X in the output $\tilde{F}(X)$. Therefore, the data of $\operatorname{im}_{\mathrm{ob}} \tilde{F}$ and $\operatorname{im}_{\mathrm{mor}} \tilde{F}$ constitutes a subcategory of Set. It remains to show that $\tilde{F}$ is an isomorphism onto its image.

To be precise, let $\Phi\colon \mathcal{C} \to \tilde{F}(\mathcal{C})$ be the functor defined $\Phi(X) \coloneqq \tilde{F}(X)$ and $\Phi(f) = \tilde{F}(f)$ for every $X \in \operatorname{ob}\mathcal{C}$ and $f \in \operatorname{mor}\mathcal{C}$. We claim that Φ is an isomorphism. Indeed, we may define a functor $\Psi\colon \tilde{F}(\mathcal{C}) \to \mathcal{C}$ as follows. For every $X \in \operatorname{ob}\tilde{F}(\mathcal{C})$, we may write $X = \tilde{F}(Y)$ for some unique $Y \in \operatorname{ob}\mathcal{C}$ and thus we may define $\Psi(X) \coloneqq Y$. Similarly, for any $f \in \operatorname{mor}\tilde{F}(\mathcal{C})$, there is exactly one morphism $g \in \operatorname{mor}\mathcal{C}$ that maps to f under $\tilde{F}$, and we may define $\Psi(f) \coloneqq g$. Now it is immediate that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity functors. □

As we alluded to previously, it is rare to encounter a category that is not concretizable. By Theorem 1.2.0.1, this means that most categories that are commonly encountered are isomorphic to some subcategory of Set, so the objects of most of our categories may be interpreted as sets and the morphisms may be interpreted as functions between those sets. However, we will eventually encounter a category that is not concretizable.

It turns out that for many applications, the notion of isomorphism of categories is too restrictive to be useful. It is often more interesting to consider the following weaker notion of *equivalence*. To define this notion, we will need to define the notion of a *natural transformation*, which will be related to the morphisms we constructed via commuting squares in Example 1.1.1.10.

Definition 1.2.0.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be covariant functors. A **natural transformation** $\mathscr{T}\colon F \to G$ consists of the data of a morphism $\mathscr{T}_X\colon F(X) \to G(X)$ for every $X \in \operatorname{ob}\mathcal{C}$. such that for every morphism $f\colon X \to Y$ in $\operatorname{mor}\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\mathscr{T}_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\mathscr{T}_Y"'] & G(Y) \end{tikzcd} \tag{1.2.0.3}$$

A natural transformation $\mathscr{T}\colon F \to G$ is a **natural isomorphism** if the morphism $\mathscr{T}_X$ is an isomorphism for every $X \in \operatorname{ob}\mathcal{C}$. We denote F and G being naturally isomorphic by writing $F \cong G$. ◇

Natural transformations can be thought of as morphisms between *functors*, with a natural isomorphism between two functors implying that the functors carry “essentially” the same data. With this interpretation in mind, we define equivalence of categories.

Definition 1.2.0.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An **equivalence** of the categories $\mathcal{C}$ and $\mathcal{D}$ is the data of two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to the identity functor on $\mathcal{C}$ and $F \circ G$ is naturally isomorphic to the identity functor on $\mathcal{D}$. ◇

This is of course a weaker notion than isomorphism of categories as given in Definition 1.2.0.4. We are usually concerned with equivalences of categories as opposed to isomorphisms between categories since they appear much more readily throughout mathematics. This is part of larger philosophy which we will see reappear several times: that in the categorical viewpoint we rarely care for objects up to strict equality—we usually only care about objects up to isomorphism.

In line with this philosophy, we may say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if every object of $\mathcal{D}$ is isomorphic to some object in $\operatorname{im}_{\operatorname{ob}} F$. Using this notion, we may characterize when a functor gives an equivalence of categories.

Theorem 1.2.0.2. *A covariant functor induces an equivalence of categories if and only if it is full, faithful, and essentially surjective.* ◇

Proof. In one direction, suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a covariant functor inducing an equivalence of categories. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor so that the compositions $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors. Let $\epsilon\colon F \circ G \to \mathbf{1}_{\mathcal{D}}$ and $\eta\colon G \circ F \to \mathbf{1}_{\mathcal{C}}$ be natural isomorphisms.

For any $X \in \operatorname{ob}\mathcal{D}$, note that ϵ gives the data of an isomorphism $\epsilon_X\colon (F \circ G)(X) \to X$. In particular, note that X is isomorphic to some object in the object-image of F. Hence, F is essentially surjective.

Next, suppose $X, Y \in \operatorname{ob}\mathcal{C}$ and consider morphisms $f, g \in \operatorname{mor}_{\mathcal{C}}(X, Y)$ with $F(f) = F(g)$. Then, $G(F(f)) = G(F(g))$, so the following diagram commutes.

$$\begin{tikzcd} X \arrow[d, "f"'] & (G \circ F)(X) \arrow[l, "\eta_X"'] \arrow[r, "\eta_X"] \arrow[d, "(G\circ F)(f)=(G\circ F)(g)"'] & X \arrow[d, "g"] \\ Y & (G \circ F)(Y) \arrow[l, "\eta_Y"] \arrow[r, "\eta_Y"'] & Y \end{tikzcd} \tag{1.2.0.4}$$

By the uniqueness result of Proposition 1.2.0.2 and Diagram 1.2.0.4, it follows that $f = g$. Hence, F is a faithful functor. An analogous argument shows that G is a faithful functor.

Now, suppose that $h \in \operatorname{mor}_{\mathcal{D}}(F(X), F(Y))$. Note that by Proposition 1.2.0.2, there exists a morphism $j\colon X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
(G \circ F)(X) \arrow[r, "\eta_X"] \arrow[d, "G(h)"'] & X \arrow[d, dashed, "j"] \\
(G \circ F)(Y) \arrow[r, "\eta_Y"'] & Y
\end{tikzcd} \tag{1.2.0.5}$$

Now, observe that since η is a natural isomorphism between $\mathbf{1}_{\mathcal{C}}$ and $G \circ F$, the following diagram will also commute.

$$\begin{tikzcd}
(G \circ F)(X) \arrow[r, "\eta_X"] \arrow[d, "(G\circ F)(j)"'] & X \arrow[d, "j"] \\
(G \circ F)(Y) \arrow[r, "\eta_Y"'] & Y
\end{tikzcd} \tag{1.2.0.6}$$

But by the uniqueness result of Proposition 1.2.0.2 along with Diagram 1.2.0.5 and Diagram 1.2.0.6, it follows that $G(h) = (G \circ F)(j) = G(F(j))$. Since G is faithful, it follows that $h = F(j)$, so F is full. This completes the first direction,

In the other direction, suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a full, faithful, and essentially surjective functor. For each $X \in \operatorname{ob}\mathcal{D}$, since F is essentially surjective there exists some $G(X) \in \operatorname{ob}\mathcal{C}$ such that $F(G(X)) \cong X$. For each $X \in \operatorname{ob}\mathcal{D}$ let $\epsilon_X\colon F(G(X)) \to X$ be an isomorphism. Now by Proposition 1.2.0.2, any morphism $f \in \operatorname{mor}\mathcal{D}$ will induce a unique morphism $\tilde{f}$ such that the following diagram commutes.

$$\begin{tikzcd}
F(G(\operatorname{dom} f)) \arrow[r, "\epsilon_{\operatorname{dom} f}"] \arrow[d, dashed, "\tilde{f}"'] & \operatorname{dom} f \arrow[d, "f"] \\
F(G(\operatorname{codom} f)) \arrow[r, "\epsilon_{\operatorname{codom} f}"'] & \operatorname{codom} f
\end{tikzcd} \tag{1.2.0.7}$$

Since F is full, there exists a morphism $G(f)\colon G(\operatorname{dom} f) \to G(\operatorname{codom} f)$ such that $\tilde{f} = F(G(f))$. Since F is faithful, this morphism $G(f)$. Since F is faithful, this morphism $G(f)$ is unique. We claim our constructions of $G(X)$ and $G(f)$ together carry the data of a functor $G\colon \mathcal{D} \to \mathcal{C}$. Indeed, suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are arbitrary morphisms in $\operatorname{mor}\mathcal{D}$. Then on one hand, the following diagram commutes by construction.

$$\begin{tikzcd}
F(G(X)) \arrow[r, "\epsilon_X"] \arrow[d, dashed, "F(G(g\circ f))"'] & X \arrow[d, "g\circ f"] \\
F(G(Z)) \arrow[r, "\epsilon_Z"'] & Z
\end{tikzcd} \tag{1.2.0.8}$$

On the other hand, note that we may apply the construction of Diagram 1.2.0.7 twice to obtain the following commutative diagram.

$$\begin{tikzcd}
F(G(X)) \arrow[r, "\epsilon_X"] \arrow[d, dashed, "F(G(f))"'] & X \arrow[d, "f"] \\
F(G(Y)) \arrow[r, "\epsilon_Y"'] \arrow[d, dashed, "F(G(g))"'] & Y \arrow[d, "g"] \\
F(G(Z)) \arrow[r, "\epsilon_Z"'] & Z
\end{tikzcd} \tag{1.2.0.9}$$

Ignoring the morphism $\epsilon_Y : F(G(Y)) \to Y$ we may focus on the commutativity of the outer rectangle in Diagram 1.2.0.9. This outer rectangle is essentially Diagram 1.2.0.8 with the morphism $F(G(g)) \circ F(G(f))$ in place of $F(G(g \circ f))$. By the uniqueness result of Proposition 1.2.0.2, it follows that

$$F(G(g \circ f)) = F(G(g)) \circ F(G(f)) = F(G(g) \circ G(f)), \tag{1.2.0.10}$$

where the last equality follows from the functoriality of F. Now by the faithfulness of F and Equation 1.2.0.10, we have that $G(g \circ f) = G(g) \circ G(f)$. So G is functorial. Moreover, note that the following diagram commutes for $\ell = F(G(\mathbf{1}_X))$ by construction and $\ell = \mathbf{1}_{F(G(X))} = F(\mathbf{1}_{G(X)})$.

$$\begin{tikzcd}
F(G(X)) \arrow[r, "\epsilon_X"] \arrow[d, "\ell"'] & X \arrow[d, "\mathbf{1}_X"] \\
F(G(Y)) \arrow[r, "\epsilon_X"'] & X
\end{tikzcd} \tag{1.2.0.11}$$

By the uniqueness of Proposition 1.2.0.2, we have $F(G(\mathbf{1}_X)) = F(\mathbf{1}_{G(X)})$. The faithfulness of F then implies that $G(\mathbf{1}_X) = \mathbf{1}_{G(X)}$. We have now shown that $G : \mathcal{D} \to \mathcal{C}$ is a functor. Note that it is immediate from the construction of the morphisms ϵ_X that they constitute a natural isomorphism $\epsilon : F \circ G \to \mathbf{1}_{\mathcal{D}}$.

For each $X \in \operatorname{ob}\mathcal{C}$, note that $\epsilon_{F(X)} : (F \circ G)(F(X)) \to F(X)$ is an isomorphism. Hence, we have morphisms $\epsilon^{-1}_{F(X)} : F(X) \to (F \circ G)(F(X))$. Since F is full and faithful, there exists a unique morphism $\eta_X : X \to G(F(X))$ such that $F(\eta_X) = \epsilon^{-1}_{F(X)}$. Note that for any morphism $f : X \to Y$, we will have $F(f) \circ \epsilon_{F(X)} = \epsilon_{F(Y)} \circ (F \circ G)(F(f))$ by the naturality of ϵ. This rearranges to $\epsilon^{-1}_{F(Y)} \circ F(f) = (F \circ G)(F(f)) \circ \epsilon^{-1}_{F(X)}$, and the faithfulness of F allows us to conclude then that

$$\eta_Y \circ f = (G \circ F)(f) \circ \eta_X. \tag{1.2.0.12}$$

Hence, the isomorphisms η_X assemble to give us a natural isomorphism $\eta : G \circ F \to \mathbf{1}_{\mathcal{C}}$. We have thus shown that F induces an equivalence of categories. □

In category theory, we most often only care for constructions that are invariant under equivalence of categories, rather than isomorphism of categories. Note also that in the proof of Theorem 1.2.0.2, we heavily relied on the properties of fully faithful functors. Such functors, which establish bijections between morphism-classes, are very important. The Yoneda embeddings are special examples of such functors and we will study them in the next section.

1.2.1 Examples

Example 1.2.1.1. In Example 1.1.1.2 we mentioned that the category of affine schemes and the opposite category of the category of commutative rings are "equivalent". Indeed, they are equivalent in the sense of Definition 1.2.0.6. One functor that gives part of this equivalence was discussed in Example 1.1.1.2: it is the contravariant functor $\operatorname{Spec} : \mathsf{CRing}^{\mathrm{op}} \to \mathsf{ASch}$ that takes commutative rings to their spectra and ring homomorphisms to the induced maps between spectra. We need a functor going the other way to establish equivalence. This functor is the *global sections functor* $\Gamma\colon \mathsf{ASch} \to \mathsf{CRing}^{\mathrm{op}}$ that takes every affine scheme to the ring of global sections of its structure sheaf and takes every morphism of affine schemes to the corresponding map on the global sections of the structure sheaves. It is then a standard result of algebraic geometry that $\operatorname{Spec}\circ\Gamma$ and $\Gamma\circ\operatorname{Spec}$ are naturally isomorphic to the identity functors. $\diamond$

Example 1.2.1.2. Let k be a field. For each positive integer n, let us fix a basis $\mathcal{B}_n$ of the vector space k^n. We define a functor $\Phi\colon \mathsf{Mat}_k \to \mathsf{FinDimVect}_k$ as follows. For every $n \in \operatorname{ob}\mathsf{Mat}_k$, define $\Phi(n) := k^n$. For every morphism $f \in \operatorname{mor}\mathsf{Mat}_k$, define $\Phi(f)$ to be the linear transformation $k^{\operatorname{dom} f} \to k^{\operatorname{codom} f}$ represented by the matrix f (if $\operatorname{dom} f$ and $\operatorname{codom} f$ are positive) with respect to the bases $\mathcal{B}_{\operatorname{dom} f}$ and $\mathcal{B}_{\operatorname{codom} f}$. The crowning theorem of elementary linear algebra is that the functor Φ induces an equivalence of categories. This equivalence of categories allows us to study abstract finite dimensional vector spaces living in $\mathsf{FinDimVect}_k$ by doing concrete matrix calculations in Mat_k. The fact that these two categories are equivalent is traditionally proven directly with elementary techniques in any first course on linear algebra, but for us it is an immediate consequence of Theorem 1.2.0.2 applied to the functor Φ. $\diamond$

Example 1.2.1.3. Let k be a field. Recall in Example 1.1.1.7 we discussed dualization and double dualization endofunctors of the category Vect_k. We claim that the dualization functor is not natural in the sense that it is not naturally isomorphic to the identity endofunctor. Indeed, one obstruction is immediate: the identity functor is covariant while the dualization functor is contravariant. Nonetheless, one may relax the definition of natural transformation to allow for the comparison of covariant and contravariant functors. In this sense, a "generalized natural transformation" between the identity and dualization functors exists if for every k-linear map $f\colon V \to W$, there exist isomorphisms $\mathscr{T}_V\colon V \to V^*$ and $\mathscr{T}_W\colon W \to W^*$ such that the following diagram commutes in Vect_k:

$$\begin{tikzcd} V \arrow[r, "\mathscr{T}_V"] \arrow[d, "f"'] & V^* \\ W \arrow[r, "\mathscr{T}_W"'] & W^* \arrow[u, "f^*"'] \end{tikzcd} \tag{1.2.1.1}$$

This is still impossible in the category Vect_k, since an infinite-dimensional vector space is not isomorphic to its dual. So as a last resort, we can ask if morphisms $\mathscr{T}_V$ and $\mathscr{T}_W$ such that Diagram 1.2.1.1 commutes in $\mathsf{FinDimVect}_k$. The answer is still no: if f is the zero map and V is a vector space with dimension at least 1, then $f^* \circ \mathscr{T}_W \circ f$ is the zero map while $\mathscr{T}_V$ cannot be.

Diagram 1.2.1.1 is a special case of a more general notion of a natural transformation known as a *dinatural transformation*. The failure of the dualization functor to be dinaturally isomorphic to the identity functor on $\mathsf{FinDimVect}_k$ is a consequence of the fact that an isomorphism between a finite-dimensional vector space and its dual is equivalent to a choice of basis for the vector space—and no such choice can be said to be "natural" (at least any more than any other choice of basis). This is the sense in which we say that a finite-dimensional vector space is isomorphic to its dual, but not canonically or naturally so.

On the other hand, the double dualization functor is indeed naturally isomorphic to the identity functor in the category $\mathsf{FinDimVect}_k$. Indeed, we can explicitly give an isomorphism between a vector space and its double dual without having to choose a basis (unlike the situation of a vector space and its dual). Let V be a finite-dimensional k-vector space and let V^{**} be its double dual. We may define an "evaluation map" $\mathscr{E}_V\colon V \to V^{**}$ given by $[\mathscr{E}_V(v)](\alpha) := \alpha(v)$ for all $\alpha \in V^*$. Note that $\mathscr{E}_V$ is injective since its kernel is the intersection of the kernels of all elements in V^* which is clearly trivial. Since V is finite-dimensional, $\dim_k V = \dim_k V^{**}$ so this map is surjective as well (if we do not assume that V is finite-dimensional, this evaluation map still gives us coordinate-free way to at least embed V in V^{**}).

Let $f\colon V \to W$ be an arbitrary morphism in $\mathsf{FinDimVect}_k$. Note then that for arbitrary $v \in V$ and $\psi \in W^*$, we can compute

$$[(f^{**} \circ \mathscr{E}_V)(v)](\psi) = [\mathscr{E}_V(v)](\psi \circ f) = (\psi \circ f)(v) = \psi(f(v)) = [(\mathscr{E}_W \circ f)(v)](\psi). \tag{1.2.1.2}$$

Hence, the following diagram commutes in $\mathsf{FinDimVect}_k$:

$$\begin{tikzcd}
V \arrow[r, "\mathscr{E}_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\
W \arrow[r, "\mathscr{E}_W"'] & W^{**}
\end{tikzcd} \tag{1.2.1.3}$$

Therefore, the isomorphisms of the form $\mathscr{E}_V$ are the components of a natural isomorphism $\mathscr{E}$ between the identity functor and the double dualization functor on $\mathsf{FinDimVect}_k$. $\diamond$

Example 1.2.1.4. These ideas appear in analytic contexts too. The *Riesz representation theorem* has many different forms, but one version can be expressed as follows.

1.3 The Yoneda Lemma

We define representable functors, which will be central in our discussion of the Yoneda lemma.

Definition 1.3.0.1. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is

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1.5.1 Examples

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