Hodge Theory and Complex Algebraic Geometry

Andrew Paul

1 Complexification of the Real Tangent Space

We recall the notion of *extension of scalars* from algebra. Let R and S be rings (with unity) and let $f\colon R \to S$ be a ring homomorphism. Suppose that M is an R-module. We may construct a natural S-module N along with an injection $M \hookrightarrow N$ by noting first that the homomorphism f induces an R-module structure on S (where $r \cdot s \coloneqq f(r)s$ for all $r \in R$ and $s \in S$). Hence, we may define $N \coloneqq M \otimes_R S$. On the pure tensors $m \otimes s \in N$, we define scalar multiplication by an element $t \in S$ via $t \cdot (m \otimes s) \coloneqq m \otimes ts$, and extend this by linearity to all elements of N. M naturally injects into N via the map $m \mapsto m \otimes 1$.

Given a real vector space V, the *complexification* of V is the extension of scalars $V \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the inclusion map $\mathbb{R} \hookrightarrow \mathbb{C}$. Note that we may identify the copy of V living naturally inside of $V \otimes_{\mathbb{R}} \mathbb{C}$ as the subspace of fixed points of the conjugation endomorphism on $V \otimes_{\mathbb{R}} \mathbb{C}$, which is defined on pure tensors via $v \otimes z \mapsto v \otimes \overline{z}$.

Given a positive integer n and a $2n$-dimensional differentiable real manifold M, we may complexify each tangent space $T_p(M)$ to obtain $T_p(M) \otimes_{\mathbb{R}} \mathbb{C}$. We claim that this complexification can be identified with the set of complex derivations of smooth functions $M \to \mathbb{C}$ (where $\mathbb{C}$ is endowed with its usual smooth structure—the one on $\mathbb{R}^2$).

Let $C^\infty_{\mathbb{C}}$ be the sheaf of smooth complex-valued functions on M and let $(C^\infty_{\mathbb{C}})_p$ denote its stalk at the point $p \in M$. Let us denote the space of complex derivations of $(C^\infty_{\mathbb{C}})_p$ by $\mathrm{Der}_{\mathbb{C}}\left((C^\infty_{\mathbb{C}})_p\right)$.

Proposition 1.0.1. *On an even-dimensional real manifold M, there exists a complex vector space isomorphism $T_p(M) \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{Der}_{\mathbb{C}}\left((C^\infty_{\mathbb{C}})_p\right)$.*

Proof. We will define a map $\Phi\colon T_p(M) \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{Der}_{\mathbb{C}}\left((C^\infty_{\mathbb{C}})_p\right)$ by defining it on the pure tensors and then extending it via linearity. Note first that every $[f] \in (C^\infty_{\mathbb{C}})_p$ represented by some function f, decomposes as $f = g + ih$ for smooth real-valued functions g and h. Now for $v \in T_p(M)$ and $z \in \mathbb{C}$, we may define

$$[\Phi(v \otimes z)]([f]) \coloneqq z(v([g]) + iv([h])) \tag{1.0.1}$$

Indeed, this is a derivation. To show this, it suffices to prove that the image of each pure tensor is a derivation, since any sum of derivations is a derivation. If $f_1 = g_1 + ih_1$ and $f_2 = g_2 + ih_2$ are smooth complex-valued functions in a neighborhood of p. Let $v \otimes z$ be an arbitrary pure tensor in $T_p(M) \otimes_{\mathbb{R}} \mathbb{C}$. We compute

$$\begin{aligned}
[\Phi(v \otimes z)]([f_1 f_2]) &= [\Phi(v \otimes z)]([g_1 g_2 - h_1 h_2 + i(g_1 h_2 + g_2 h_1)]) \\
&= z\left(v([g_1 g_2 - h_1 h_2]) + iv([g_1 h_2 + g_2 h_1])\right) \\
&= z\big(g_1(p)v([g_2]) + g_2(p)v([g_1]) - h_1(p)v([h_2]) - h_2(p)v([h_1]) \\
&+ ig_1(p)v([h_2]) + ih_2(p)v([g_1]) + ig_2(p)v([h_1]) + ih_1(p)v([g_2])\big) \\
&= z\left[(g_1(p) + ih_1(p))(v([g_2]) + iv([h_2])) + (g_2(p) + ih_2(p))(v([g_1]) + iv([h_1]))\right] \\
&= f_1(p)[\Phi(v \otimes z)]([f_2]) + f_2(p)[\Phi(v \otimes z)]([f_1]).
\end{aligned} \tag{1.0.2}$$

The $\mathbb{C}$-linearity of Φ is easy to check. Suppose now that some element $(v \otimes 1) + (w \otimes i)$ is in the kernel of Φ. This means that for every smooth real-valued function g and h, we have

$$v([g]) + iv([h]) + iw([g]) - w([h]) = 0. \tag{1.0.3}$$

Letting g be a constant function and h vary over all functions in Equation 1.0.3, we find that $v = w = 0$. Therefore, Φ is injective.

A calculation similar to Equation 1.0.2 involving a check of the Leibniz rule shows that given $\varphi \in \mathrm{Der}_{\mathbb{C}}\left((C^\infty_{\mathbb{C}})_p\right)$, we will have that $\mathrm{Re}\left(\varphi|_{(C^\infty_{\mathbb{R}})_p}\right)$ and $\mathrm{Im}\left(\varphi|_{(C^\infty_{\mathbb{R}})_p}\right)$ can be interpreted as elements of $T_p(M) \cong \mathrm{Der}_{\mathbb{R}}\left((C^\infty_{\mathbb{R}})_p\right)$. We will use this fact to show that Φ is surjective. Let $\varphi \in \mathrm{Der}_{\mathbb{C}}\left((C^\infty_{\mathbb{C}})_p\right)$ be arbitrary and set v

be the element of $T_p(M)$ corresponding to $\operatorname{Re}\left(\varphi|_{C^\infty_{\mathbb{R}}(M)|_p}\right)$ and let w be the element of $T_p(M)$ corresponding to $\operatorname{Im}\left(\varphi|_{(C^\infty_{\mathbb{R}})_p}\right)$. Note that for any smooth, real-valued functions g and h,

$$\begin{aligned}\varphi([g+ih]) &= \varphi([g]) + i\varphi([h]) \\ &= \operatorname{Re}\varphi([g]) + i\operatorname{Im}\varphi([g]) + i\operatorname{Re}\varphi([h]) - \operatorname{Im}\varphi([h]) \\ &= [\Phi((v\otimes 1) + (w\otimes i))]([g+ih]).\end{aligned} \tag{1.0.4}$$

Hence, we have established that Φ is an isomorphism. $\square$

Despite the isomorphism indicated in Proposition 1.0.1, the complexification $T_p(M)\otimes_{\mathbb{R}}\mathbb{C}$ is not what deserves the name of "holomorphic tangent bundle" when M is a complex manifold. Instead, if M is a complex manifold, we define the holomorphic tangent space to be $\operatorname{Der}_{\mathbb{C}}(H_p)$, where H is the sheaf of holomorphic functions on M. These glue together to form what is called the holomorphic tangent bundle $T^{\mathrm{hol}}(M)$. One may see (using a proof analogous to the real case) that the holomorphic tangent bundle on a complex manifold M is generated by the operators $\frac{\partial}{\partial z_j}$. The real version of this statement involves rewriting every C^∞ function using Taylor's theorem with remainder. Of course, in the complex setting there are stronger variants of Taylor's theorem with remainder given by Cauchy's integral formula.

We will show that for a complex manifold M containing p, the holomorphic tangent space $\operatorname{Der}_{\mathbb{C}}(H_p)$ can be naturally identified with a subspace of the complexification $T_p(M)\otimes_{\mathbb{R}}\mathbb{C}$ of the real tangent space. In fact, this subspace can be defined if M has just an *almost complex structure*, which is in general weaker than a complex structure. The question of when an almost complex structure can be promoted to a complex structure will be answered later by the Newlander-Nirenberg theorem.

Definition 1.0.1. Let V be a real vector space and let M be a differentiable manifold. An **almost complex structure** on V is an endomorphism $J\colon V\to V$ such that $J\circ J = -\mathbf{1}_V$. An **almost complex structure** on M is an endomorphism I of the tangent bundle $T(M)$ such that $I\circ I = -\mathbf{1}_{T(M)}$. The pair (M, I) where M is a differentiable manifold and I is an almost complex structure on M is called an **almost complex manifold**.

Observe that if (M, I) is an almost complex manifold, and $p\in M$, we have that $I_p^2 = -\mathbf{1}_{T_p(M)}$ by definition. Taking determinants, this implies that $0 < (\det I_p)^2 = (-1)^{\dim_{\mathbb{R}}(M)}$. Hence, $\dim_{\mathbb{R}}(M)$ must be even.

Note that if M is a complex manifold, then given a holomorphic local chart $\varphi\colon U\to\mathbb{C}^n$, we may identify the real tangent bundle $T(U)$ with the space $U\times\mathbb{C}^n$. Hence, we may define the endomorphism $I_U\colon T(U)\to T(U)$ to take the action of $\mathbf{1}_U\times\iota\colon U\times\mathbb{C}^n$ once we pass to the aforementioned identification, where $\mathbf{1}_U\colon U\to U$ is the identity and $\iota\colon\mathbb{C}^n\to\mathbb{C}^n$ is the scalar multiplication by i map. One may check that that this map I_U is compatible with the map I_V where $V\to\mathbb{C}^n$ is another holomorphic local chart, in the sense that $I_U|_{T(U\cap V)} = I_V|_{T(U\cap V)}$. Hence, these maps on open subsets of the tangent bundle $T(U)$ glue together to a map I on the full tangent bundle $T(M)$ that clearly satisfies $I\circ I = -\mathbf{1}_{T(M)}$. So every complex manifold inherits a natural almost complex structure.

Let (M, I) be an almost complex manifold. Since $I\circ I = -\mathbf{1}_{T(M)}$, on each tangent space $T_p(M)$ we have that $I_p\colon T_p(M)\to T_p(M)$ is a real-linear map whose minimal polynomial is x^2+1. Hence, the eigenvalues of I_p are i and $-i$, which are not real. Therefore, to speak of eigenvectors, we must extend our scalars and work in the complexification $T_p(M)\otimes_{\mathbb{R}}\mathbb{C}$. This promotes I_p to a complex-linear map $\tilde{I}_p\colon T_p(M)\otimes_{\mathbb{R}}\mathbb{C}\to T_p(M)\otimes_{\mathbb{R}}\mathbb{C}$. In the complexification, since the minimal polynomial does not have any repeating roots, we have an eigenbasis of $T_p(M)\otimes_{\mathbb{R}}\mathbb{C}$ with respect to the map $\tilde{I}_p$. The eigenvalues of $\tilde{I}_p$ are i and $-i$, and we denote the eigenspace corresponding to the eigenvalue i by $T_p^{1,0}(M)$ and the eigenspace corresponding to the eigenvalue $-i$ by $T_p^{0,1}(M)$. These eigenspaces glue together to form subbundles $T^{1,0}(M)$ and $T^{0,1}(M)$ of the complexified tangent bundle $T(M)\otimes_{\mathbb{R}}\mathbb{C}$.

Lemma 1.0.1. *Suppose V is a real vector space of real dimension $2n$ for some positive integer n and let J be an almost complex structure on V. The complex vector space V^J defined by $i\cdot v := J(v)$ for all $v\in V_{\mathbb{C}}$ has complex dimension n.*

Proof. Let $\{e_k\}_{k\in S}$ be a basis of V^J. Note that $ie_k \neq e_j$ for any $k, j \in S$, since the e_k are all $\mathbb{C}$-linearly independent. We just need to observe that $\{e_k\}_{k\in S} \cup \{ie_k\}_{k\in S}$ is a basis of V. Indeed, note that these vectors are $\mathbb{R}$-linearly independent, since the e_k are $\mathbb{C}$-linearly independent. Moreover, they clearly $\mathbb{R}$-span V. Hence, $2|S| = 2n$ so $\dim_{\mathbb{C}} V^J = |S| = n$. $\square$

Proposition 1.0.2. *Let M be a complex manifold (and thus also an almost complex manifold). Then, $T^{\text{hol}}(M) \cong T^{1,0}(M) \cong T(M)^J$ as smooth vector bundles.*

Proof. Pick $p \in M$. Let J be the almost complex structure on M induced by its complex structure. Pick a basis $v_1, v_2, \ldots, v_n$ of the eigenspace $T_p^{1,0}(M)$. Define the map $\Psi_p : T_p^{1,0}(M) \to T_p^{0,1}(M)$ by

$$\Psi_p(v \otimes 1 + v \otimes i) := v \otimes 1 + v \otimes i \tag{1.0.5}$$

We may endow $T_p(M)$ with a complex vector space structure by declaring $i \cdot v = J_p(v)$ for all $v \in T_p(M)$. Let $T_p(M)^J$ denote the space $T_p(M)$ endowed with this complex vector space structure. Define the map $\Phi_p : T_p(M)^J \to T_p^{1,0}(M)$ by

$$\Phi_p(v) := \frac{1}{2}[(v \otimes 1) - (J_p(v) \otimes i)]. \tag{1.0.6}$$

We claim that Φ_p is an isomorphism of complex vector spaces. First, note that it is clear that Φ_p is a complex-linear map. Moreover, for every $v \in T_p(M)^J$ we have

$$\begin{aligned}\tilde{J}_p(\Phi_p(v)) &= \frac{1}{2}[(J_p(v) \otimes 1) - (J_p^2(v) \otimes i)] \\ &= \frac{1}{2}[(J_p(v) \otimes 1) + (v \otimes i)] \\ &= \frac{i}{2}[(v \otimes 1) - (J_p(v) \otimes i)] \\ &= i\Phi_p(v).\end{aligned} \tag{1.0.7}$$

Hence, we indeed have that $\operatorname{im} \Phi_p \subseteq T_p^{1,0}(M)$. Next, note that if $\Phi_p(v) = 0$, then we have that $v \otimes 1 = J_p(v) \otimes i$. Taking conjugates, we see that $v \otimes 1 = J_p(v) \otimes -i = -(J_p(v) \otimes i) = -(v \otimes 1)$, so $v = 0$. Hence, Φ_p has trivial kernel.

Finally, note that $T_p^{1,0}(M) \cong T_p^{0,1}(M)$ via the conjugation map. Since $T_p(M) \otimes_{\mathbb{R}} \mathbb{C} = T_p^{1,0}(M) \oplus T_p^{0,1}(M)$, this implies that

$$\dim_{\mathbb{C}} T_p^{1,0}(M) = \frac{1}{2} \dim_{\mathbb{C}} (T_p(M) \otimes_{\mathbb{R}} \mathbb{C}) = \frac{1}{4} \dim_{\mathbb{R}} (T_p(M) \otimes_{\mathbb{R}} \mathbb{C}) = \frac{1}{2} \dim_{\mathbb{R}} M, \tag{1.0.8}$$

with the second equality coming from Lemma 1.0.1. Likewise, $\dim_{\mathbb{C}} T_p(M)^J \cong \frac{1}{2} \dim_{\mathbb{R}} M$ also by Lemma 1.0.1. Hence Φ_p must be surjective as well.

This establishes that $T(M)^J$ is isomorphic to $T^{1,0}(M)$ as a smooth vector bundle. Note that $T^{\text{hol}}(M)$ is generated by the derivations

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) = \frac{1}{2}\left[\left(\frac{\partial}{\partial x_j} \otimes 1\right) - \left(\frac{\partial}{\partial y_j} \otimes i\right)\right] = \Phi\left(\frac{\partial}{\partial x_j}\right). \tag{1.0.9}$$

Note that in the last equality of Equation 1.0.9 follows from the fact that $J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$, since multiplication by i in a tangent space amounts to a rotation by $\frac{\pi}{2}$. Since the $\frac{\partial}{\partial x_j}$ generate $T_p(M)^J$ and Φ is an isomorphism, we conclude that $T^{\text{hol}}(M) \cong T^{1,0}(M)$ as smooth vector bundles. $\square$

2 The Newlander-Nirenberg Theorem